

MST121 Chapter C3



The Open  
University

A first level  
interdisciplinary  
course

Using  
**Mathematics**

CHAPTER

**C3**

**BLOCK C**

**CONTINUOUS MODELS**

# *Differential equations and modelling*







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interdisciplinary  
course

# Using Mathematics

CHAPTER

C3

## BLOCK C

### CONTINUOUS MODELS

# *Differential equations and modelling*

*Prepared by the course team*



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The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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# Contents

Study guide	4
Introduction	5
1 Differential equations and direct integration	6
1.1 Differential equations and solutions	6
1.2 Direct integration	11
2 Solution by separation of variables	15
2.1 Further use for the Chain Rule	15
2.2 Separation of variables	18
3 Modelling growth and decay	26
3.1 Radioactive decay	26
3.2 Continuous model for population growth	31
4 Differential equations with the computer	38
5 A look ahead	40
Summary of Chapter C3	41
Learning outcomes	41
Summary of Block C	42
Solutions to Activities	43
Solutions to Exercises	49
Index	52



# Study guide

There are five sections in this chapter. They are intended to be studied consecutively in four study sessions. Section 4 requires the use of the computer and Computer Book C.

The pattern of study for each session might be as follows.

Study session 1: Section 1.

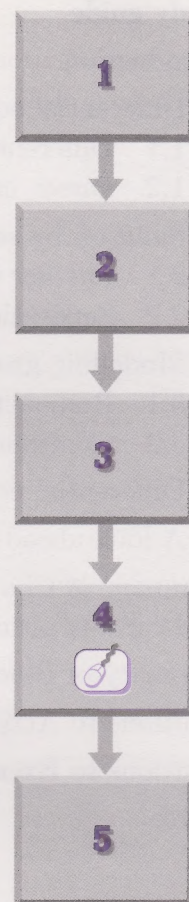
Study session 2: Section 2.

Study session 3: Section 3.

Study session 4: Section 4 and Section 5.

Each session requires two to three hours.

Sections 2 and 3 may take longer to study than the other sections. Section 5 is short and will not be assessed.





# Introduction

In many modelling situations, the ultimate mathematical aim is to obtain an explicit relationship between two variables. This involves expressing the dependent variable, which we denote here by the symbol  $y$ , as a function  $F$  of an independent variable, denoted by  $x$ . However, it may not be possible to write down an equation of the form  $y = F(x)$  straight away. This desired outcome may be achievable only by solving some other type of equation.

Often the equation that we have to solve involves not only the variables  $x$  and  $y$  themselves but also derivatives of  $y$  with respect to  $x$ . You saw examples of such situations in Chapter C2, Section 3, in the context of modelling motion. Equations that include derivatives, such as

$$\frac{dy}{dx} = 2y \quad \text{and} \quad \frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 9x^2y = \sin(2x),$$

are called *differential equations*. Any function  $y = F(x)$  which satisfies such an equation is called a *solution* of the differential equation. For example, the function  $y = e^{2x}$  is a solution of the first differential equation displayed above.

The *order* of a differential equation is the order of the highest derivative that appears in it. So the first equation displayed above is an example of a *first-order* differential equation, since it contains only the first derivative of the dependent variable with respect to the independent variable. A *second-order* differential equation, such as the second equation displayed above, contains a second derivative, and maybe a first derivative as well, but no higher derivatives.

This chapter concentrates exclusively on first-order differential equations, and only on particular types of these. Section 1 deals with differential equations that can be written in the form

$$\frac{dy}{dx} = f(x),$$

where  $f$  is a given function. Such equations can be solved by *direct integration* of the function  $f$ . Section 2 shows how differential equations that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

may be solved, using the method of *separation of variables*. This method again depends on being able to perform integrations.

Section 3 looks more closely at a special type of differential equation soluble by separation of variables, namely  $dy/dx = Ky$ , where  $K$  is a constant. This type of equation can be used to model decay and growth processes, such as radioactive decay or the growth of a population.

Section 4 shows how the computer can be used to plot *direction fields*, which give a visual impression of the overall behaviour of solutions of a differential equation. This is followed by a demonstration of how solutions of differential equations may be obtained both numerically and graphically, using *Euler's method*. These approaches apply more widely than those considered earlier in the chapter.

Finally, Section 5 indicates briefly some directions in which the theory of differential equations can be further extended.

In these examples the independent variable was  $t$ , denoting time, while the dependent variable was  $v$  (velocity) or  $s$  (position).

In this chapter we normally use Leibniz notation for derivatives. This is common in the context of differential equations.

Higher-order differential equations are defined similarly.



# 1 Differential equations and direct integration

In Subsection 1.1 we look at first-order differential equations and their solutions. In Subsection 1.2 the method of *direct integration* is used to solve a simple type of differential equation.

## 1.1 Differential equations and solutions

We start by drawing together some of the terminology referred to already in the Introduction.

### Definitions

- ◇ A **differential equation** is an equation that relates an independent variable,  $x$  say, a dependent variable,  $y$  say, and one or more derivatives of  $y$  with respect to  $x$ .
- ◇ The **order** of a differential equation is the order of the highest derivative that appears in the equation. A **first-order** differential equation involves the first derivative,  $dy/dx$ , and no higher derivatives.
- ◇ A **solution** of a differential equation is a function  $y = F(x)$  (or a more general equation relating  $x$  and  $y$ ) for which the differential equation is satisfied.

Solutions expressed in the form of 'a more general equation relating  $x$  and  $y$ ' are discussed in Section 2.

See Chapter C2, Section 3.

As you saw in Chapter C2, a differential equation may feature symbols for the independent and dependent variables other than  $x$  and  $y$ , and you will see further examples of this later in the current chapter, when a modelling context makes it appropriate. While considering the general theory of differential equations, however, we shall stick to the use of  $x$  and  $y$ .

A first-order differential equation features the first derivative,  $dy/dx$ . The differential equation may also involve separate occurrences of  $x$  or  $y$ , or both. Thus each of the following is a first-order differential equation:

$$\frac{dy}{dx} = 5; \quad \frac{dy}{dx} = 3x^2 + 2; \quad \frac{dy}{dx} = -2y; \quad \frac{dy}{dx} = \frac{2x}{3y^2 + 1}.$$

You will shortly see methods which lead to the solutions of all these differential equations. First, however, you are asked to verify in a number of cases that a function which is claimed to be the solution of a differential equation does in fact satisfy that equation. This type of exercise is useful in checking that no error has been made in the working leading to a solution. It is similar to the practice of checking whether a calculated indefinite integral is correct by differentiating the answer obtained.

In addition, such checking will provide you with some further practice in differentiation, using Leibniz notation, including application of the Product, Quotient and Composite Rules. The following example indicates what is required.

These rules were introduced in Chapter C1, Section 4.



**Example 1.1** Verifying that a given function is a solution

In each case, show that the given function satisfies the given differential equation.

We describe each function  $F$  by writing  $y = F(x)$ .

- (a)  $y = x^3 + 2x$ ;  $\frac{dy}{dx} = 3x^2 + 2$
- (b)  $y = \frac{x}{1+x^2}$ ;  $\frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2}$
- (c)  $y = x^3 e^{2x}$ ;  $\frac{dy}{dx} = \frac{(3+2x)y}{x}$  ( $x > 0$ )
- (d)  $y = \ln(1+x^2)$ ;  $\frac{dy}{dx} = 2xe^{-y}$

**Solution**

In each case, we differentiate the given function to find an expression equivalent to the left-hand side of the differential equation, and then show that the same expression arises from the right-hand side of this equation. This verifies that the function is a solution of the differential equation.

- (a) The differential equation is  $dy/dx = 3x^2 + 2$ . The derivative of the given function,  $y = x^3 + 2x$ , is

$$\frac{d}{dx}(x^3 + 2x) = 3x^2 + 2,$$

which is the same expression as the right-hand side of the differential equation, as required.

- (b) The differential equation is  $dy/dx = (1-x^2)/(1+x^2)^2$ . In order to differentiate the given function,  $y = x/(1+x^2)$ , we apply the Quotient Rule. The function can be written as  $y = u/v$ , where

$$u = x \quad \text{and} \quad v = 1 + x^2,$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = 2x.$$

Hence we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{v^2} \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) \\ &= \frac{(1+x^2)(1) - (x)(2x)}{(1+x^2)^2} \\ &= \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}, \end{aligned}$$

which is the same expression as the right-hand side of the differential equation, as required.

- (c) The differential equation is  $dy/dx = (3+2x)y/x$ . In order to differentiate the given function,  $y = x^3 e^{2x}$ , we apply the Product Rule. The function can be written as  $y = uv$ , where

$$u = x^3 \quad \text{and} \quad v = e^{2x},$$

for which

$$\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{dv}{dx} = 2e^{2x}.$$

See Chapter C1, Subsection 4.2.

See Chapter C1, Subsection 4.1.



Hence we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= (3x^2)(e^{2x}) + (x^3)(2e^{2x}) = (3 + 2x)x^2e^{2x}.\end{aligned}$$

Now we substitute  $y = x^3e^{2x}$  into the right-hand side of the differential equation, to obtain

$$\frac{(3 + 2x)y}{x} = \frac{(3 + 2x)x^3e^{2x}}{x} = (3 + 2x)x^2e^{2x}.$$

This is the same expression as was obtained above for the derivative, as required.

- (d) The differential equation is  $dy/dx = 2xe^{-y}$ . In order to differentiate the given function,  $y = \ln(1 + x^2)$ , we apply the Composite (Chain) Rule. The function can be written as

$$y = \ln u, \quad \text{where } u = 1 + x^2,$$

for which

$$\frac{dy}{du} = \frac{1}{u} \quad \text{and} \quad \frac{du}{dx} = 2x.$$

Hence we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \frac{1}{u}(2x) = \frac{2x}{1 + x^2}.\end{aligned}$$

Now we substitute  $y = \ln(1 + x^2)$  into the right-hand side of the differential equation, to obtain

$$\begin{aligned}2xe^{-y} &= 2x \exp(-\ln(1 + x^2)) \\ &= 2x \exp(\ln((1 + x^2)^{-1})) \\ &= 2x(1 + x^2)^{-1} = \frac{2x}{1 + x^2}.\end{aligned}$$

This is the same expression as was obtained above for the derivative, as required.

Here are some similar cases for you to try.

### Activity 1.1 Verifying that a given function is a solution

In each case, show that the given function satisfies the given differential equation.

(a)  $y = 5\sqrt{x}$ ;  $\frac{dy}{dx} = \frac{5}{2\sqrt{x}} \quad (x > 0)$

(b)  $y = \sin(e^{3x})$ ;  $\frac{dy}{dx} = 3e^{3x} \cos(e^{3x})$

(c)  $y = e^x \cos(2x)$ ;  $\frac{dy}{dx} = y(1 - 2 \tan(2x)) \quad (-\frac{1}{4}\pi < x < \frac{1}{4}\pi)$

(d)  $y = \frac{1-x}{1+x}$ ;  $\frac{dy}{dx} = -\frac{1}{2}(y+1)^2$

Solutions are given on page 43.

The restriction  $x > 0$  in the statement of part (c) ensures that division by  $x$  is valid here. Another suitable restriction is  $x < 0$ .

See Chapter C1, Subsection 4.3.

Recall that  $\exp a$  is an alternative (and in this case convenient) way of writing  $e^a$ , and that  $\exp$  is the inverse function of  $\ln$ .



In Example 1.1(a), we showed that the function  $y = x^3 + 2x$  is a solution of the differential equation  $dy/dx = 3x^2 + 2$  because

$$\frac{d}{dx}(x^3 + 2x) = 3x^2 + 2.$$

However, this is not the only solution of the differential equation. Since

$$\frac{d}{dx}(x^3 + 2x + c) = 3x^2 + 2,$$

where  $c$  is any constant, any function of the form  $y = x^3 + 2x + c$  is also a solution of the differential equation  $dy/dx = 3x^2 + 2$ . We say that  $y = x^3 + 2x + c$ , where  $c$  is an arbitrary constant, is the *general solution* of this differential equation, since it describes the whole infinite family of possible solutions. By contrast, the function obtained when a particular value is chosen for  $c$ , as with the original function  $y = x^3 + 2x$  (where  $c = 0$ ), is called a *particular solution* of the differential equation.

You may recall that an arbitrary constant also arises when finding the indefinite integral of a function and, as you will see shortly (or may already have noticed), there is a very close link between integration and the process of solving differential equations.

For a discussion of integrals and arbitrary constants, see Chapter C2, Section 1.

The general solution of a first-order differential equation will usually feature an arbitrary constant. This does not always occur in the form of a '+c' term tagged on at the end of the solution formula, but that will be the pattern for the differential equations solved in this section.

#### General and particular solutions

- ◇ The **general solution** of a differential equation is the set of all possible solutions of the equation. It usually involves one or more arbitrary constants.
- ◇ A **particular solution** of a differential equation is a single solution of the equation, which consists of a relationship between the dependent and independent variables that contains no arbitrary constant.

In many cases, where a differential equation arises in a mathematical model, the value of the dependent variable is known for one value of the independent variable. For example, in Chapter C2 the motion of a parachutist was modelled by a differential equation for  $dv/dt$ , where  $v$  is velocity and  $t$  is time. In addition, a value of  $v$  was specified at time  $t = 0$ . This permitted an appropriate value to be chosen for the arbitrary constant that arose from integration, and hence led to a single function (a particular solution) which described the velocity of the parachutist.

See Chapter C2, Example 3.1.

Similar considerations apply more generally. A first-order differential equation has a general solution and, in order for a particular solution to be obtained from this, a further condition on the solution is needed. This often takes the form of an *initial condition*, in which a value of the dependent variable  $y$  is specified at a given value of  $x$ . When taken together, the differential equation and initial condition are called an *initial-value problem*.



The second notation stems from the common practice of writing

$$y = y(x),$$

in which the symbol  $y$  does 'double duty', as a variable on the left-hand side and as a function on the right. This is convenient because it reduces the number of separate symbols that might otherwise be required. Nor does it cause any ambiguity, since the context determines which of the two uses is intended.

This is revision of what you saw in Chapter C2, Example 3.1, Activity 3.1 and Subsection 3.2.

### Initial-value problem

- ◇ An **initial condition** associated with a first-order differential equation requires that the dependent variable  $y$  takes a specified value,  $b$  say, when the independent variable  $x$  has a given value,  $a$  say. This is often written as

$$y = b \text{ when } x = a, \quad \text{or as} \quad y(a) = b.$$

The numbers  $a$  and  $b$  are called **initial values** for  $x$  and  $y$ , respectively.

- ◇ The combination of a first-order differential equation and an initial condition is called an **initial-value problem**. The solution of an initial-value problem is a particular solution of the differential equation which also satisfies the initial condition.

The following example and activity concern finding a particular solution from the general solution of a differential equation, using an initial condition.

### Example 1.2 Applying an initial condition

As you have seen, the general solution of the differential equation  $dy/dx = 3x^2 + 2$  is  $y = x^3 + 2x + c$ , where  $c$  is an arbitrary constant. Find the particular solution that satisfies the initial condition

$$y = 9 \text{ when } x = 2; \quad \text{that is,} \quad y(2) = 9.$$

#### Solution

We substitute the values  $x = 2$  and  $y = 9$  (simultaneously) into the expression  $y = x^3 + 2x + c$ . This gives

$$9 = 2^3 + 2 \times 2 + c = 12 + c,$$

so  $c = -3$ . The required particular solution is therefore

$$y = x^3 + 2x - 3.$$

Here is a similar problem for you to try.

### Activity 1.2 Applying an initial condition

The general solution of the differential equation

$$\frac{dy}{dx} = \frac{5}{2\sqrt{x}} \quad (x > 0)$$

is  $y = 5\sqrt{x} + c$ , where  $c$  is an arbitrary constant. Find the particular solution that satisfies the initial condition

$$y = 7 \text{ when } x = 1; \quad \text{that is,} \quad y(1) = 7.$$

A solution is given on page 43.



## 1.2 Direct integration

In the remainder of this section we concentrate on differential equations which can be written in the form

$$\frac{dy}{dx} = f(x). \quad (1.1)$$

As you saw in Chapter C2, differentiation is ‘undone’ by integration, and so the general solution of the differential equation (1.1) is given by the indefinite integral of the function  $f(x)$ ; that is,

See Chapter C2, Section 1.

$$y = \int f(x) dx.$$

### Example 1.3 Finding the general and a particular solution

- (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = x^2.$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $y = 2$  when  $x = 1$ .

#### Solution

- (a) The general solution of the differential equation is given by

$$y = \int x^2 dx.$$

The general solution is therefore

$$y = \frac{1}{3}x^3 + c,$$

where  $c$  is an arbitrary constant.

(It is always possible, and advisable, to check a general solution by differentiating and substituting back into the differential equation. Here  $y = \frac{1}{3}x^3 + c$ , so  $dy/dx = x^2$ , as required.)

- (b) Putting  $x = 1$  and  $y = 2$  into the general solution, we obtain

$$2 = \frac{1}{3}(1)^3 + c.$$

Hence  $c = \frac{5}{3}$ , and the required particular solution is

$$y = \frac{1}{3}x^3 + \frac{5}{3}.$$

The method used in Example 1.3 is called the **direct integration** method for solving differential equations. It can be applied to any differential equation, such as  $dy/dx = x^2$ , for which the derivative of the dependent variable is equal to a known function of the independent variable.



### Direct integration

- ◇ The general solution of the differential equation

$$\frac{dy}{dx} = f(x) \quad (1.1)$$

is the indefinite integral

$$y = \int f(x) dx = F(x) + c,$$

where  $F(x)$  is any integral of  $f(x)$  and  $c$  is an arbitrary constant.

- ◇ Any initial condition

$$y = b \text{ when } x = a, \quad \text{that is, } y(a) = b,$$

enables a value for the arbitrary constant  $c$  to be found. The corresponding particular solution satisfies both the differential equation and the initial condition.

Recall that  $F(x)$  is an integral of  $f(x)$  if

$$\frac{d}{dx}(F(x)) = f(x).$$

Hence  $y = F(x)$  is a particular solution of the differential equation.

The method can also be applied, of course, if symbols other than  $x$  and  $y$  are used for the independent and dependent variables. Any differential equation of the form

$$\frac{d(\text{dependent variable})}{d(\text{independent variable})} = f(\text{independent variable})$$

We say 'in principle' because in practice it may not be possible to find explicitly an integral of  $f$ . However, you will not meet such cases in this text.

can in principle be solved using direct integration. For example, the differential equation

$$\frac{dq}{dt} = 4t + 3$$

has the general solution

$$q = \int (4t + 3) dt = 2t^2 + 3t + c,$$

where  $c$  is an arbitrary constant.

### Activity 1.3 Can direct integration be applied?

You are *not* asked to solve these differential equations.

To which of the following differential equations can the direct integration method be applied?

- (a)  $\frac{dy}{dx} = x \cos x$     (b)  $\frac{dx}{dt} = x \cos x$     (c)  $\frac{dx}{dt} = t \cos t$   
 (d)  $\frac{dy}{dx} = y \cos y$     (e)  $\frac{dx}{dy} = y \cos y$     (f)  $\frac{dx}{dt} = x \cos t$

Solutions are given on page 43.



**Activity 1.4 Applying direct integration**

- (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{2x}.$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition
- $y = 2$
- when
- $x = 0$
- .

Solutions are given on page 44.

In Activity 1.4 you saw that the general solution of the differential equation  $dy/dx = e^{2x}$  is  $y = \frac{1}{2}e^{2x} + c$ , where  $c$  is an arbitrary constant. Some graphs corresponding to this family of solutions are shown in Figure 1.1. Each of these solution curves has the property that, at any point  $(x, y)$  on it, the gradient of the curve is  $dy/dx = e^{2x}$ . All the solution curves in Figure 1.1 are parallel, in the sense that the vertical separation of any two curves is a constant, independent of  $x$ . This is a property of all differential equations of the type  $dy/dx = f(x)$ . However, as you will see in the rest of the chapter, it is *not* a general property of first-order differential equations.

**Activity 1.5 Applying direct integration again**

Solve the initial-value problem

$$\frac{dx}{dt} = \sin t, \quad x = 1 \text{ when } t = 0.$$

A solution is given on page 44.

**Activity 1.6 Mass of a burning rocket**

A toy rocket consists of a casing and fuel. Initially the total mass of the rocket is 200 grams. The fuel is ignited at time  $t = 0$  and then burns in such a way that the rocket's total mass  $m$  grams at time  $t$  seconds satisfies the differential equation

$$\frac{dm}{dt} = -8t \quad (t > 0).$$

When the fuel is exhausted, the residual mass of the rocket casing is 100 grams.

- Find the general solution of the given differential equation.
- Find the particular solution of the differential equation that satisfies the initial condition  $m = 200$  when  $t = 0$ .
- Hence find the time at which the fuel is exhausted; that is, find the time  $t$  when  $m = 100$ , for the particular solution obtained in part (b).

Solutions are given on page 44.

You can check the general solution by differentiating it and then substituting back into the differential equation.

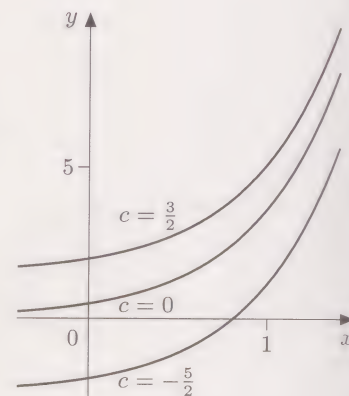


Figure 1.1 Graphs of  $y = \frac{1}{2}e^{2x} + c$ , for  $c = -\frac{5}{2}$ , 0 and  $\frac{3}{2}$

In this activity the independent variable is  $t$  and the dependent variable is  $x$ .

The derivative  $dm/dt$  represents the rate of change of mass. This quantity is negative because the mass decreases as the fuel is burnt.



## Summary of Section 1

This section has introduced:

- ◇ first-order differential equations, and the solutions of such equations (general and particular);
- ◇ the process of obtaining a particular solution from a general solution, using an initial condition;
- ◇ the method of direct integration, to solve differential equations of the form  $dy/dx = f(x)$ .

## Exercises for Section 1

### Exercise 1.1

Show that the function  $y = -1/x$  ( $x > 0$ ) is a solution of the differential equation  $dy/dx = y^2$ .

### Exercise 1.2

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = \sqrt{x} \quad (x > 0).$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $y = 5$  when  $x = 4$ .
- (c) Hence find the value of  $y$  when  $x = 25$ .

### Exercise 1.3

Solve each of the following initial-value problems.

- (a)  $\frac{du}{dx} = \cos(2x)$ ,  $u = -2$  when  $x = \frac{1}{4}\pi$
- (b)  $\frac{dx}{dt} = \frac{1}{t}$  ( $t > 0$ ),  $x = 3$  when  $t = 1$

### Exercise 1.4

A large spherical snowball melts in such a way that, at any instant, the rate of decrease of its volume  $V$  is proportional to its surface area  $A$ , with constant of proportionality  $k > 0$ . Initially the snowball has a radius of 100 cm. After 2 days the snowball has lost half its initial volume.

Using the formulas  $V = \frac{4}{3}\pi r^3$  and  $A = 4\pi r^2$  for the volume and surface area of a sphere, in terms of its radius  $r$ , it can be shown that the change in radius  $r$  (in cm) of the snowball with time  $t$  (in days since melting started) is modelled by the differential equation

$$\frac{dr}{dt} = -k \quad (r \geq 0).$$

- (a) Find the general solution of this differential equation.
- (b) Find the particular solution of the differential equation that satisfies the given initial condition.
- (c) Use the condition that the snowball has lost half its initial volume after 2 days to find the value of the constant of proportionality  $k$ .
- (d) How long will it take for the snowball to disappear completely?

You are asked to establish this differential equation in Exercise 2.4.



## 2 Solution by separation of variables

In Subsection 1.2 you saw that the general solution of the differential equation

$$\frac{dy}{dx} = f(x)$$

is given by the indefinite integral

$$y = \int f(x) dx.$$

In this section we develop a method for finding solutions of a further type of differential equation, namely, those of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where  $f$  and  $g$  are known functions. This method, called *separation of variables*, is described in Subsection 2.2. As a preliminary, Subsection 2.1 shows how the Composite (Chain) Rule for differentiation may usefully be employed even when one of the component functions in the composite is not known explicitly.

### 2.1 Further use for the Chain Rule

In Section 1 you saw the solutions of several differential equations. In each case, the solution was specified in the form  $y = F(x)$ , for some function  $F$ . When solving a differential equation, it is sometimes not possible to reach such an explicit form for the solution function. For example, suppose that it has been deduced, from a differential equation for  $dy/dx$ , that the variables  $x$  and  $y$  are related by the equation

$$y^3 + y = x^2 + 9. \quad (2.1)$$

It is not straightforward to solve this equation for  $y$  and hence to reach the form  $y = F(x)$  for the solution. However, equation (2.1) does describe a solution, even if only implicitly. For example, a corresponding solution curve can be plotted, and the value of  $dy/dx$  at a point  $(x, y)$  on this curve is the slope of the curve at that point.

More generally, suppose that the solution of a differential equation is obtained in the form

$$H(y) = F(x), \quad (2.2)$$

where  $H$  and  $F$  are known functions, and  $H(y) \neq y$ . (Equation (2.1) is of this form.) This is called an **implicit solution** of the differential equation. By contrast, a solution written in the form  $y = F(x)$ , where  $F$  is a known function, is called the **explicit solution** of the differential equation. In seeking to solve a differential equation, it is the explicit solution which we seek. However, if this is impossible to find, then we must settle for an implicit solution.

In Subsection 1.1 you saw how to verify that a given function is an explicit solution of a specified differential equation. In order to check that a calculated implicit solution satisfies the differential equation from which it was obtained, the Chain Rule is employed. This is possible because, in equation (2.2), the  $y$  that appears on the left-hand side is (implicitly) a

The possibility of a solution being 'a more general equation relating  $x$  and  $y$ ' was allowed for in the definition of a solution of a differential equation in the box on page 6.

See Example 1.1 and Activity 1.1.



function of  $x$ . (All the implicit solutions we deal with have the form of equation (2.2) and are such that  $y$  is a function of  $x$ .) If we write

$$z = H(y) = F(x),$$

then two expressions can be deduced for  $dz/dx$  as follows. Since  $z = F(x)$ , we have

$$\frac{dz}{dx} = F'(x).$$

Since also  $z = H(y)$ , where  $y$  is a function of  $x$ , we have, by the Chain Rule,

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = H'(y) \frac{dy}{dx}.$$

(This step is called **implicit differentiation** of  $H(y)$  with respect to  $x$ .)

Equating these two expressions for  $dz/dx$ , we see that

$$\text{if } H(y) = F(x), \quad \text{then} \quad H'(y) \frac{dy}{dx} = F'(x). \quad (2.3)$$

The next example indicates how this result can be put to use.

### Example 2.1 Verifying a solution given implicitly

- (a) Show that equation (2.1),

$$y^3 + y = x^2 + 9,$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1}.$$

- (b) What is the slope of the corresponding solution curve at the point  $(1, 2)$ ?

#### Solution

- (a) We have  $H(y) = F(x)$ , where  $H(y) = y^3 + y$  and  $F(x) = x^2 + 9$ . It follows from result (2.3) that

$$\frac{d}{dy}(y^3 + y) \frac{dy}{dx} = \frac{d}{dx}(x^2 + 9),$$

which leads to

$$(3y^2 + 1) \frac{dy}{dx} = 2x.$$

Dividing through by  $3y^2 + 1$  gives

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1},$$

as required.

- (b) At  $(1, 2)$ , the slope of the curve is

$$\frac{dy}{dx} = \frac{2 \times 1}{3 \times 2^2 + 1} = \frac{2}{13}.$$

Here are similar problems for you to try.



**Activity 2.1 Verifying a solution given implicitly**

(a) Show that the equation

$$y + \sin y = x + e^x - 1$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{1 + e^x}{1 + \cos y} \quad (-\pi < y < \pi).$$

(b) What is the slope of the corresponding solution curve at the point  $(0, 0)$ ?

Solutions are given on page 44.

**Activity 2.2 Another verification**

In Chapter C1 you saw that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Use this result, and the trigonometric identity  $\sec^2 \theta = 1 + \tan^2 \theta$ , to show that the equation

$$\tan y = x$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{1 + x^2} \quad \left(-\frac{1}{2}\pi < y < \frac{1}{2}\pi\right).$$

A solution is given on page 44.

**Comment**

In fact, it is possible in this case to write the solution explicitly. For if  $\tan y = x$  and  $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ , then  $y = \arctan x$ . Hence the result of this activity shows that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.$$

It can be shown in a similar way that

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}} \quad (-1 < x < 1).$$

See Chapter C1, Activity 4.4(a). The same differentiation appears in Chapter C2, Activity 2.4(a).

This trigonometric identity was introduced in Chapter C2, Subsection 2.1.

This derivative formula is valid for all values of  $x$ .

The previous example and activities indicate the usefulness of implicit differentiation using the Chain Rule, as described by result (2.3). In the next subsection, you will see that result (2.3) enables us to explain a solution process for a wider class of differential equations than those considered in Subsection 1.2.



## 2.2 Separation of variables

We now derive a method for solving differential equations of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where  $f$  and  $g$  are known functions. The starting point for explaining this method is the result, obtained in Subsection 2.1, that

$$\text{if } H(y) = F(x), \quad \text{then } H'(y)\frac{dy}{dx} = F'(x). \quad (2.3)$$

Integration is the reverse process to differentiation. Hence, faced with a differential equation of the form

$$h(y)\frac{dy}{dx} = f(x),$$

where  $f$  and  $h$  are known functions, a solution will be given by the equation  $H(y) = F(x)$ , provided that

$$H'(y) = h(y) \quad \text{and} \quad F'(x) = f(x).$$

It follows that

$$\text{if } h(y)\frac{dy}{dx} = f(x), \quad \text{then} \quad \int h(y) dy = \int f(x) dx. \quad (2.4)$$

The latter equation gives the general solution of the differential equation, in implicit form.

The next example shows how this result may be applied.

### Example 2.2 Finding a general solution in implicit form

- (a) Find, in implicit form, the general solution of the differential equation

$$2y\frac{dy}{dx} = 3x^2.$$

- (b) Find the corresponding particular solution that satisfies the initial condition  $y = 3$  when  $x = 2$ .

#### Solution

- (a) Comparing the differential equation with result (2.4), we have  $h(y) = 2y$  and  $f(x) = 3x^2$ . Hence the general solution of the differential equation is given by

$$\int 2y dy = \int 3x^2 dx; \quad \text{that is, } y^2 + a = x^3 + b,$$

where  $a$  and  $b$  are arbitrary constants. These two constants can be combined to form the single arbitrary constant  $c = b - a$ . Thus the general solution, in implicit form, is

$$y^2 = x^3 + c,$$

where  $c$  is an arbitrary constant.

- (b) To apply the initial condition, put  $x = 2$  and  $y = 3$  into the general solution. This gives

$$3^2 = 2^3 + c,$$

so  $c = 1$ . Hence the required particular solution is

$$y^2 = x^3 + 1.$$

The phrase ‘in implicit form’ is used to describe a general or particular solution that is an implicit solution. The phrase ‘in explicit form’ is applied similarly.

This demonstrates that, when both sides of an equation are integrated, we need introduce only one constant of integration. We shall follow this practice from here on.



Here is a similar problem for you to try.

### Activity 2.3 Finding a general solution

- (a) Find, in implicit form, the general solution of the differential equation

$$\frac{1}{y^2} \frac{dy}{dx} = -1 \quad (y > 0).$$

- (b) Find the corresponding explicit form of this general solution.

Solutions are given on page 45.

The approach demonstrated in Example 2.2 and Activity 2.3 can be extended to solve any differential equation of the form

$$\frac{dy}{dx} = f(x)g(y). \quad (2.5)$$

A differential equation of this type is said to be **separable**, and the corresponding method of solution is called **separation of variables**. To solve equation (2.5), we first rewrite it so that there is a function of  $x$  alone on the right-hand side. This is done by dividing both sides by  $g(y)$ , to give

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

Now we can apply result (2.4), with  $h(y) = 1/g(y)$ . This shows that the general solution of equation (2.5), in implicit form, is

$$\int \frac{1}{g(y)} dy = \int f(x) dx. \quad (2.6)$$

To summarise, the method of separation of variables is as follows.

#### Separation of variables

The method applies to differential equations of the form

$$\frac{dy}{dx} = f(x)g(y). \quad (2.5)$$

1. Divide both sides by  $g(y)$ , to obtain

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

2. Integrate both sides with respect to  $x$ . From result (2.4), the outcome is

$$\int \frac{1}{g(y)} dy = \int f(x) dx. \quad (2.6)$$

3. Carry out the two integrations, introducing *one* arbitrary constant, to obtain the general solution in implicit form. If possible, manipulate the resulting equation to make  $y$  the subject, thus expressing the general solution in explicit form.

(A general solution, in implicit or explicit form, can be checked by substituting back into the differential equation.)

We assume here that any value  $y$  for which  $g(y) = 0$  is excluded from consideration.

At this stage the variables are separated, with a function of  $y$  alone on the left-hand side and a function of  $x$  alone on the right-hand side.



The combination of a differential equation and an initial condition is called an *initial-value problem* – see page 10.

Any initial condition provides a value for the arbitrary constant in step 3. The corresponding particular solution satisfies both the differential equation and the initial condition.

You might like to note that the equation in step 2 can be written down quickly from the differential equation obtained in step 1 by regarding  $dy/dx$  temporarily as an ordinary quotient and ‘multiplying through by  $dx$ ’ before adding the two integral signs. This shortcut is a convenient aid in recalling this step of the method. However, remember that the derivative  $dy/dx$  is not  $dy$  divided by  $dx$ . We have not assigned a meaning to either  $dx$  or  $dy$  on their own; they make sense only when combined in a derivative such as  $dy/dx$ , or when preceded by an integral sign as in  $\int \dots dx$ .

This method can be applied also when independent and dependent variables other than  $x$  and  $y$  are used, provided that the derivative is equal to a product of a function of the independent variable and a function of the dependent variable.

**Activity 2.4 Can separation of variables be applied?**

To which of the following differential equations can the separation of variables method be applied; that is, which of them are of the form

$$\frac{d(\text{dependent variable})}{d(\text{independent variable})} = f(\text{independent variable})g(\text{dependent variable})$$

for some choice of the functions  $f$  and  $g$ ? For each of the equations below that is of this form, identify the functions  $f$  and  $g$ .

- (a)  $\frac{dy}{dx} = x \cos y$       (b)  $\frac{dx}{dt} = t \cos t$       (c)  $\frac{dp}{dt} = p \cos p$   
 (d)  $\frac{dv}{dy} = v + y$       (e)  $\frac{dv}{dy} = v + vy$       (f)  $\frac{dy}{dx} = \cos(xy)$

Solutions are given on page 45.

Parts (b) and (c) of Activity 2.4 illustrate that both the equations

$$\frac{dy}{dx} = f(x) \tag{1.1}$$

and

$$\frac{dy}{dx} = g(y) \tag{2.7}$$

Some differential equations of the type given by equation (2.7) will be considered in Section 3.

are special cases of the equation  $dy/dx = f(x)g(y)$ , with  $g(y) = 1$  in the first case and  $f(x) = 1$  in the second. As you saw in Subsection 1.2, the general solution of equation (1.1) can be found by direct integration, to which separation of variables reduces when  $g(y) = 1$ .

**Example 2.3 Applying separation of variables**

- (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{-y} \cos x.$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $y = 1$  when  $x = 0$ .



**Solution**

- (a) The differential equation is of the form of equation (2.5) with  $f(x) = \cos x$  and  $g(y) = e^{-y}$ . Dividing both sides of the differential equation by  $e^{-y}$  gives

$$e^y \frac{dy}{dx} = \cos x.$$

From this we obtain, using result (2.4),

$$\int e^y dy = \int \cos x dx.$$

On performing the integrations and including a single arbitrary constant  $c$ , this becomes

$$e^y = \sin x + c,$$

which is an implicit form of the general solution.

To make  $y$  the subject of this equation, recall that  $\ln(e^y) = y$ . Hence, on taking the natural logarithm of each side of the equation, we obtain

$$y = \ln(\sin x + c),$$

where  $c$  is an arbitrary constant. This is the explicit form of the general solution.

- (b) Putting  $x = 0$  and  $y = 1$  into the explicit form of the general solution, we have

$$1 = \ln(\sin 0 + c) = \ln c.$$

Hence  $c = e^1 = e$ , and the required particular solution is

$$y = \ln(e + \sin x).$$

Other choices for  $f(x)$  and  $g(y)$  are possible, for example,  $f(x) = 2 \cos x$  and  $g(y) = \frac{1}{2}e^{-y}$ , but all such choices lead to the same general solution.

Since  $1/e^{-y} = e^y$ , dividing by  $e^{-y}$  is the same as multiplying by  $e^y$ .

Recall that  $\ln(e) = 1$ .

**Comment**

1. The solution could now be checked, though the details are not given here.
2. In part (b) the explicit form of the general solution was used in finding the required particular solution. This particular solution can also be obtained by using the implicit general solution.

Putting  $x = 0$  and  $y = 1$  into

$$e^y = \sin x + c,$$

we obtain

$$e = \sin 0 + c,$$

so  $c = e$ . Hence the particular solution in implicit form is

$$e^y = e + \sin x,$$

and taking the natural logarithm of each side gives

$$y = \ln(e + \sin x),$$

as before.

Unless you are directed otherwise, you should find particular solutions from the explicit form of the general solution.

Here are some similar problems for you to try.

### Activity 2.5 Applying separation of variables

- (a) Find the general solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \quad (y > 0).$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $y = 1$  when  $x = 0$ .

Solutions are given on page 45.

### Activity 2.6 Applying separation of variables again

- (a) Find the general solution of the differential equation

$$\frac{dx}{dt} = t \sec x \quad \left(-\frac{1}{2}\pi < x < \frac{1}{2}\pi\right).$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $x = 0$  when  $t = 0$ .

Solutions are given on page 46.

A possible snag in applying the separation of variables method was mentioned in a margin note on page 19. We can divide both sides of the equation

$$\frac{dy}{dx} = f(x)g(y)$$

by  $g(y)$  only if  $g(y) \neq 0$ , since division by zero is an undefined operation. It is therefore necessary to exclude from consideration any value of  $y$ , the dependent variable, for which  $g(y) = 0$ . This may be done by imposing a suitable condition on  $y$  from the outset, as we did in Activity 2.5 by assuming that  $y > 0$ .

As well as having to impose restrictions on the dependent variable in order to apply the separation of variables method, we must be aware of other pitfalls that can arise. The solution obtained may be valid only for a certain range of values of the independent variable. For example, in Activity 2.5 you found that the required particular solution of the given differential equation  $dy/dx = -x/y$ , assuming that  $y > 0$ , was

$$y = \sqrt{1 - x^2}.$$

The expression  $\sqrt{1 - x^2}$  is a real number only for values of  $x$  such that  $-1 \leq x \leq 1$ . Since  $x = 1$  and  $x = -1$  both give  $y = 0$ , which is excluded by the restriction  $y > 0$  in the question, the domain of this particular solution is the interval  $(-1, 1)$ . Similar considerations apply to the corresponding general solution in Activity 2.5, which was

$$y = \sqrt{b - x^2}.$$

This formula provides positive values for  $y$  only when the arbitrary constant  $b$  is positive, and then only when  $-b^{1/2} < x < b^{1/2}$ . So the interval  $(-b^{1/2}, b^{1/2})$ , where  $b > 0$ , is the domain of this solution.



These examples show that consideration of the appropriate domain for a solution can become a bit complicated. This should not be a cause for your concern, as you will not be expected to derive the appropriate domain for a solution in the manner indicated above. However, it is as well to be aware that in practice the choice of domain may require some thought. Deriving a formula  $y = F(x)$  (or  $H(y) = F(x)$ ) for a solution does not guarantee that the formula obtained makes sense for all real numbers  $x$ , nor that the function  $F$  actually satisfies the differential equation without some restriction being imposed on its domain. Also, as has been pointed out several times, it is wise to check a proposed solution by substituting it into the differential equation.

The final activity of this section investigates a model for the motion of a space probe launched from Earth. The modelling theme is continued in the next section, where each model features a differential equation that can be solved by separation of variables.

### Activity 2.7 Motion of a space probe

A motorless space probe is launched vertically upwards from the Earth's surface. When the probe is at a distance  $r$  from the centre of the Earth, its outward velocity  $v$  satisfies the differential equation

$$\frac{dv}{dr} = -\frac{k}{vr^2} \quad (r > R),$$

where  $R$  is the radius of the Earth and  $k$  is a positive constant. All the variables and constants are measured in the appropriate SI units.

- (a) Motion of the probe outwards from the Earth corresponds to  $v > 0$ . Find the general solution of the differential equation in this case.
- (b) The initial condition corresponding to a launch velocity of  $v_R$  is

$$v(R) = v_R; \quad \text{that is, } v = v_R \text{ when } r = R.$$

Show that the corresponding particular solution for the motion outwards from Earth is

$$v = \sqrt{v_R^2 + 2k \left( \frac{1}{r} - \frac{1}{R} \right)}.$$

- (c) Show that at very large distances from the Earth, the velocity of the space probe tends towards the limiting value

$$\sqrt{v_R^2 - \frac{2k}{R}}.$$

Hence, by considering the cases  $v_R > \sqrt{2k/R}$  and  $v_R < \sqrt{2k/R}$ , explain the significance of the particular launch velocity

$$v_R = \sqrt{\frac{2k}{R}}.$$

Solutions are given on page 46.

The restriction  $r > R$  arises because the differential equation is a suitable model only when the space probe is above the surface of the Earth.

Do not spend too long on part (c) if you find it difficult.

## Summary of Section 2

This section has introduced:

- ◇ the contrast between the explicit solution of a differential equation, in the form  $y = F(x)$ , and an implicit solution, in the form  $H(y) = F(x)$ ;
- ◇ implicit differentiation of a function  $H(y)$  with respect to  $x$ , where  $y$  is a function of  $x$  whose rule is not known explicitly;
- ◇ the method of separation of variables, to solve differential equations of the form  $dy/dx = f(x)g(y)$ .

## Exercises for Section 2

### Exercise 2.1

- (a) Show that the equation

$$y + \ln y = (1 - 2x)e^{3x}$$

is an implicit solution of the differential equation

$$\frac{dy}{dx} = \frac{y(1 - 6x)e^{3x}}{y + 1} \quad (y > 0).$$

- (b) What is the slope of the corresponding solution curve at the point  $(0, 1)$ ?

### Exercise 2.2

- (a) Find the general solution of the differential equation

$$u^2 \frac{du}{dx} = x^4.$$

- (b) Find the particular solution of this differential equation that satisfies the initial condition  $u = 1$  when  $x = 1$ .

### Exercise 2.3

Solve in explicit form each of the following initial-value problems.

- (a)  $\frac{dy}{dx} = -y^3$  ( $y > 0$ ),  $y = \frac{1}{2}$  when  $x = 0$
- (b)  $\frac{dx}{dt} = tx$  ( $x > 0$ ),  $x = 2$  when  $t = 0$

### Exercise 2.4

In this exercise you are asked to establish the differential equation

$$\frac{dr}{dt} = -k \quad (r \geq 0)$$

which was solved in Exercise 1.4. The equation models the change in radius  $r$  (in cm) of a snowball with time  $t$  (in days since melting started).

The snowball melts in such a way that, at any instant, the rate of decrease of its volume  $V = \frac{4}{3}\pi r^3$  is proportional to its surface area  $A = 4\pi r^2$ , with constant of proportionality  $k > 0$ . Use these formulas for  $V$  and  $A$ , and the Chain Rule, to establish the differential equation.



**Exercise 2.5**

This exercise concerns a mathematical model for the flow of water from a tank, of constant horizontal cross-section, through a small hole near its base. According to the model, the height  $h$  of the water surface above the hole is related to the time  $t$  since the hole was uncovered. This relationship is described by the differential equation

$$\frac{dh}{dt} = -kh^{1/2} \quad (h > 0),$$

where  $k$  is a positive constant. (The value of  $k$  depends on both the area of the hole and the horizontal cross-sectional area of the water column in the tank.)

- (a) Find the general solution of this differential equation.
- (b) Find the particular solution for which  $h = h_0$  at  $t = 0$ . (Here  $h_0$  is the height of the water column above the hole at the moment when the hole is uncovered.)
- (c) Show that, according to this model, the tank empties to the level of the hole after a time  $2h_0^{1/2}/k$ .

One application of this model is to the water emptying from a vertically-sided bath via the plug-hole.

### 3 Modelling growth and decay

In this section we investigate mathematical models that are based on first-order differential equations of the form

$$\frac{dy}{dx} = Ky,$$

where  $K$  is a constant. This type of equation is soluble by separation of variables. The resulting behaviour is that of exponential growth or decay.

Subsection 3.1 concerns radioactive decay, while Subsection 3.2 returns to the topic of population change.

#### 3.1 Radioactive decay

As you may know, the nuclei of radioactive substances disintegrate spontaneously, at a rate which varies according to the substance concerned. For example, a gram of pure uranium is composed of an enormous number (about  $2.5 \times 10^{21}$ ) of uranium atoms. Each of these atoms consists of a central nucleus surrounded by a cloud of 92 electrons. There are three different kinds of naturally-occurring uranium nuclei, called uranium-234, uranium-235 and uranium-238, but uranium as mined is more than 99% uranium-238. (The numbers 234, 235 and 238 give the atomic mass of these three kinds of uranium.) From time to time one of the uranium-238 nuclei disintegrates, emitting an electrically charged particle, called an alpha particle, which can be detected using a Geiger counter. As a result of these disintegrations, the uranium slowly decays into another substance, namely, thorium-234. The amount of uranium contained in a lump of matter therefore decreases steadily with time.

Each nucleus of uranium-238 has a known probability of decay in a given time interval. However, because of the enormous number of nuclei present in even one gram of uranium-238, it is sensible to use a continuous model to describe this situation, measuring the amount of uranium in grams, rather than using a discrete model, where we would measure the number of uranium nuclei. The same applies to the decay of other radioactive substances.

As each nucleus has a constant probability of decay in a given time interval, we assume for the continuous model that the rate of decay of a radioactive substance is proportional to the amount of that substance which is present. Consequently, the change in mass  $m$  of the radioactive substance that is present at time  $t$  can be modelled by the first-order differential equation

$$\frac{dm}{dt} = -km \quad (m > 0), \tag{3.1}$$

where  $k$  is a positive constant, called the **decay constant**. The negative sign in this differential equation reflects that we have a *decaying* process, where the mass  $m$  is a decreasing function of time, so that  $dm/dt$  is negative. The condition  $m > 0$  corresponds to the fact that decay can occur only when some of the radioactive substance is present. Physically, we would expect equation (3.1) to be a good model for the situation, except where there are only a small number of radioactive nuclei, for which a discrete model would be more appropriate.

In Chapter B3, Subsection 4.1, a particle was defined as a material object whose size and internal structure may be neglected. The usage here is taken from physics, where ‘particle’ means an object which is finite but very small.

If time  $t$  is measured in days, then the unit of  $k$  is  $\text{day}^{-1}$ .



To find a particular solution of equation (3.1), we require an initial condition. If there is an amount  $m_0$  of the radioactive substance present at time  $t = 0$ , then the initial condition is

$$m = m_0 \text{ when } t = 0; \quad \text{that is, } m(0) = m_0. \quad (3.2)$$

The next activity asks you to solve the initial-value problem given by equations (3.1) and (3.2).

### Activity 3.1 Solving the radioactive decay problem

- Use separation of variables to find the general solution of equation (3.1), giving your answer in explicit form.
- Find the corresponding particular solution which satisfies the initial condition (3.2).

Solutions are given on page 46.

In the above activity you showed that the initial-value problem that models radioactive decay,

$$\frac{dm}{dt} = -km \quad (m > 0), \quad m = m_0 \text{ when } t = 0,$$

has solution

$$m = m_0 e^{-kt}. \quad (3.3)$$

The graph of this solution, shown in Figure 3.1, illustrates that the solution has the characteristics we would expect of radioactive decay. The mass of the substance decreases, initially quite quickly and then more slowly, until eventually there is practically none of the substance left.

Different radioactive substances have different values for the decay constant  $k$ . The larger the value of  $k$ , the faster the substance decays. This is illustrated in Figure 3.2.

The exponential decrease of a radioactive substance has an interesting consequence in terms of the factor by which the amount of the substance declines over any fixed time interval. If we denote the mass of the radioactive substance at time  $t$  by  $m(t)$  then, according to equation (3.3), we have

$$m(t) = m_0 e^{-kt}.$$

After a further time interval of duration  $T$ , the mass of the radioactive substance has become

$$m(t+T) = m_0 e^{-k(t+T)} = m_0 e^{-kt} e^{-kT} = m(t) e^{-kT}.$$

It follows that

$$\frac{m(t+T)}{m(t)} = e^{-kT}. \quad (3.4)$$

This equation says that, at the end of a time interval of duration  $T$ , the proportion of the original amount of a radioactive substance remaining is  $e^{-kT}$ , regardless of the time  $t$  at the start of the interval.

According to equation (3.4), the time which it takes for the mass of radioactive substance to diminish to half its original amount is given by

$$\frac{1}{2} = \frac{m(t+T)}{m(t)} = e^{-kT}.$$

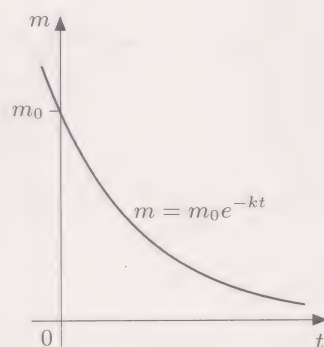


Figure 3.1 Graph of  $m = m_0 e^{-kt}$ ,  $k > 0$

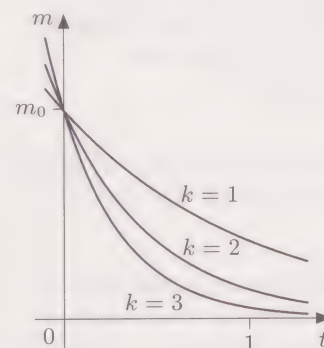


Figure 3.2 Graph of  $m = m_0 e^{-kt}$  for several values of  $k$

Here we use the rule

$$e^{x+y} = e^x e^y.$$

Taking the logarithm of each side, we obtain

$$\ln \frac{1}{2} = -kT; \quad \text{that is,} \quad kT = \ln 2.$$

Hence the mass of radioactive substance decays to half of its original amount in the time

$$T = \frac{\ln 2}{k}, \quad (3.5)$$

where  $k$  is the decay constant. This time  $T$  is called the **half-life** of the substance. Whereas the speed of decay of a radioactive substance could be described in terms of the decay constant  $k$ , it is more usual to use the half-life  $T$ . Radioactive substances have a very wide range of half-lives, from millions of years down to tiny fractions of a second.

### Activity 3.2 Using the half-life

Silicon-31 has a half-life of 2.62 hours. Find the value of the decay constant  $k$  for silicon-31. Hence calculate the proportion of a sample of silicon-31 which will still be present after six hours. Give each value to three significant figures.

A solution is given on page 47.

### Carbon dating

One useful application of radioactivity is to the dating of archaeological finds of biological origin. Carbon occurs naturally in three forms: carbon-12, carbon-13 and carbon-14. Chemically they are indistinguishable, but carbon-14 is radioactive, with a half-life of approximately 5570 years, and decays continually (into nitrogen). This decay is balanced by the production of carbon-14 in the atmosphere, and it may be assumed that the proportion of carbon-14 in naturally-occurring carbon is constant over time. This same known proportion exists in all *living* tissue.

When an organism dies and is buried, the carbon-14 is no longer replenished and the proportion of carbon-14 in the carbon decreases with time. The quantity of carbon-14 in a sample of dead tissue can be estimated from a measurement of the radioactivity, and the total quantity of carbon in the sample can be estimated from chemical analysis. These values, the known half-life of carbon-14 and the known proportion of carbon-14 in naturally-occurring carbon in living tissue form the basis for a method of estimating when the organism died. The method is known as **carbon dating**. In particular, an estimate of the proportion of the original amount of carbon-14 still present in the sample can be found.

### Activity 3.3 Using carbon dating

The bones of an animal are found in an archaeological dig. Analysis produces the estimate that 85% of the original amount of carbon-14 is still present in the bones. Taking the half-life of carbon-14 to be 5570 years, find the approximate age of these bones to three significant figures.

A solution is given on page 47.

Recall that

$$\ln \frac{1}{2} = \ln(2^{-1}) = -\ln 2.$$

You do *not* need to be able to follow the detail of the following outline description in order to attempt Activity 3.3.

In terms of numbers of atoms, this proportion is 1 to  $10^{12}$ .

This discovery was made in the 1950s by the American chemist Willard F. Libby. He was awarded the Nobel Prize for Chemistry in 1960 for his pioneering work in radiocarbon dating.



Finding the decay constant from data

You have seen that the amount  $m$  of a radioactive substance present in a lump of matter at time  $t$  is modelled by an initial-value problem whose solution is

$$m = m_0 e^{-kt}.$$

(3.3)

The decay constant  $k$  cannot be measured directly, but it can be deduced by measuring experimentally how the amount of the radioactive substance changes with time. The data in the table below were obtained by Ernest Rutherford early in the twentieth century, when he was studying radioactive decay. He observed a sample of a radioactive compound of thorium over a period of several months, taking measurements from time to time, and deducing the proportions  $m/m_0$  of the original mass of the sample which remained.

$t$ (days)	10	15	20	26	33	48	66	89
$m/m_0$	0.71	0.59	0.51	0.43	0.33	0.22	0.13	0.06

According to the model, the amount of radioactive substance present, as a proportion of its initial mass, is given by  $m/m_0 = e^{-kt}$ . Taking the logarithm of each side of this equation, we obtain

$$\ln\left(\frac{m}{m_0}\right) = -kt.$$

Hence a plot of  $\ln(m/m_0)$  against  $t$ , for the data above, should result in a straight line through the origin with slope  $-k$ . After adding to the table above a row for the values of  $\ln(m/m_0)$  to two decimal places, we obtain the table below.

$t$ (days)	10	15	20	26	33	48	66	89
$m/m_0$	0.71	0.59	0.51	0.43	0.33	0.22	0.13	0.06
$\ln(m/m_0)$	-0.34	-0.53	-0.67	-0.84	-1.11	-1.51	-2.04	-2.81

In Figure 3.3 a graph of  $\ln(m/m_0)$  against  $t$  has been plotted. It shows both the data points and a straight line through the origin which fits them well. The line is drawn through the origin because, when  $t = 0$ , we have  $\ln(m/m_0) = \ln(m_0/m_0) = \ln 1 = 0$ .

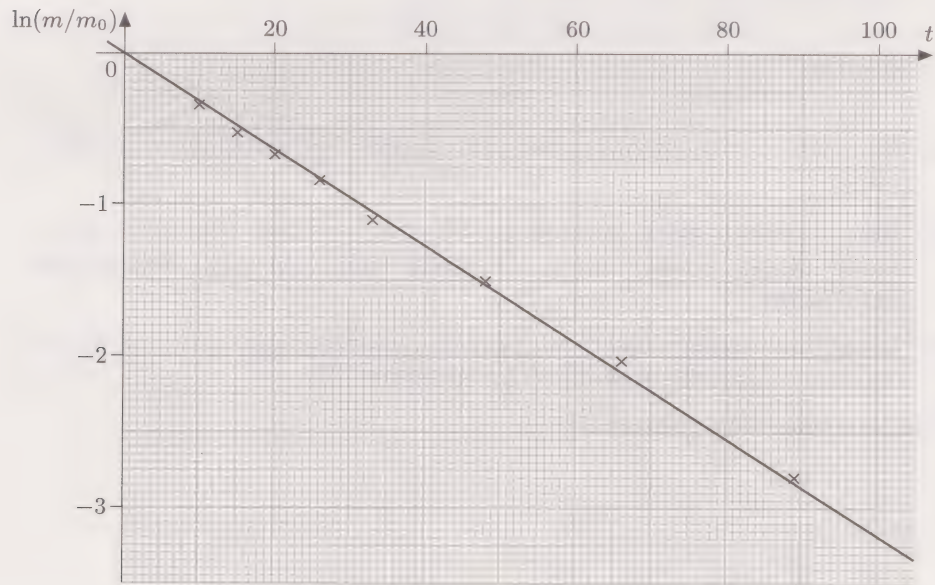


Figure 3.3 Graph of  $\ln(m/m_0)$  against  $t$  for the data in the table above

$m_0$  is the original amount of the substance.

Ernest Rutherford (1871–1937) was a New Zealand-born physicist of great distinction. He was the first to show that radioactivity is caused by the spontaneous decay of heavy atoms into slightly lighter ones, for which he was awarded the Nobel Prize for Chemistry in 1908. He also propounded the nuclear model for the atom, and became the first person to split the atom.

A straight line through the origin in the  $(x, y)$ -plane, with slope  $-k$ , has the equation  $y = -kx$ . Here we have  $t$  in place of  $x$  and  $\ln(m/m_0)$  in place of  $y$ .

In Block D you will see an objective method for calculating the line which best fits a set of data points. However, it is often sufficient to use one’s eye. This is a rather subjective approach, but a good rule of thumb is to draw the line so as to minimise the distances of the data points from the line, while ensuring that there are about the same number of data points on either side of the line. This rule of thumb is employed in Figure 3.3.

From the graph you could read the second coordinate as slightly different from  $-3.2$ .

Figure 3.3 shows that these data points are reasonably well-fitted by a straight line through the origin with negative slope. This validates use of the exponential model to describe radioactive decay. The straight line passes through the origin and the point  $(100, -3.2)$ , so its slope is approximately  $-3.2/100 = -0.032$ . Hence the decay constant for radioactive decay of the thorium compound is estimated to be

$$k = 0.032 \text{ day}^{-1}.$$

From equation (3.5), the corresponding half-life is

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.032} = 22 \text{ days (to 2 s.f.)}.$$

A number of remarks about the above estimation, the decay constant and the half-life are called for.

1. It is assumed in the above method for estimating the decay constant that *hand* plotting of the data and the graph are used.
2. The values of  $\ln(m/m_0)$  were given to two decimal places because hand plotting on a normal-sized piece of graph paper can represent this level of accuracy at most.
3. The first coordinate of the point chosen in order to calculate the slope of the line drawn by eye should be near the high end of the  $t$  data range, and ideally be arithmetically convenient. The first coordinate of the point  $(100, -3.2)$  satisfies both requirements.
4. The value of  $T$  was given to two significant figures, the same accuracy as for the value of  $k$ , estimated from the graph. Slightly different estimates for  $k$ , obtained from other lines ‘fitted by eye’, can affect the value of  $T$ . For example, the point  $(100, -3.25)$  leads to  $k = 0.0325$  and  $T = 21.3$  to three significant figures.

You should bear the above points in mind when attempting the following activity, and in similar situations.

**Activity 3.4 Finding the half-life of uranium-239**

The table below shows data for the radioactive decay of a sample of uranium-239.

$t$ (minutes)	10	20	30	40	50
$m/m_0$	0.73	0.56	0.42	0.29	0.24
$\ln(m/m_0)$					

- (a) Calculate to two decimal places the entries for the third row of this table.
- (b) Plot, on graph paper, a graph of  $\ln(m/m_0)$  against  $t$  for these data. Draw ‘by eye’ a line through the origin which you think best fits the data points.
- (c) Using your straight line from part (b), estimate values for the decay constant  $k$  and the half-life  $T$  for uranium-239.

Solutions are given on page 47.



**Comment**

As was said before, fitting a straight line to data points 'by eye' is a subjective process. This means that your values of the decay constant and half-life may not be exactly the same as those given in the solution. However, they should not be very different!

### 3.2 Continuous model for population growth

Earlier in the course you saw mathematical models, based on recurrence relations, to describe the way in which population size alters with time. These models all had the property that they predicted population size only at discrete intervals of time, typically at annual intervals. The aim here is to develop a model which describes a population at *any* instant of time, on a continuously changing basis. In practice, of course, populations are made up of individuals, and so the size of a population can take only integer values and does not change continuously. However, if the population size is large, then its behaviour may be approximated satisfactorily by a continuous function, and continuous models for populations are therefore of value.

If  $P$  represents the size of a population at time  $t$ , then the instantaneous rate of change of the population size is given by the derivative  $dP/dt$ . A simple modelling assumption is that the rate of change of the population size is proportional to the current population size. We then have the differential equation

$$\frac{dP}{dt} = KP \quad (P > 0),$$

where  $K$  is a constant, called the **proportionate growth rate**. The condition  $P > 0$  corresponds to the fact that population change can occur only for a positive population.

The value of  $K$  may be positive (birth rate exceeds death rate), negative (birth rate less than death rate) or zero (birth and death rates equal). If the initial population size (at time  $t = 0$ ) is  $P_0$ , then we have the initial-value problem

$$\frac{dP}{dt} = KP \quad (P > 0), \quad P = P_0 \text{ when } t = 0. \quad (3.6)$$

This problem has the same mathematical form as that given by equations (3.1) and (3.2) for the radioactive decay model. Hence the solution to problem (3.6) can be written down immediately, by comparison with the previous case whose solution is given by equation (3.3). The solution to problem (3.6) is therefore

$$P = P_0 e^{Kt}. \quad (3.7)$$

Figure 3.4 shows some typical graphs of this solution. The case  $K > 0$  corresponds to population growth,  $K = 0$  gives a static population size and  $K < 0$  corresponds to population decline. The behaviour for  $K < 0$  is mathematically identical to that examined earlier for radioactive decay (where  $K = -k$ ), so we shall now look at examples for which  $K > 0$ . The next activity shows how equation (3.7) can be put to use.

Discrete models for population change were developed in Chapter B1, Sections 2–4.

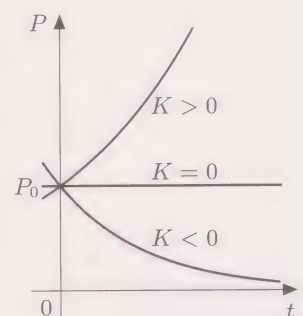


Figure 3.4 Graph of  $P = P_0 e^{Kt}$  for several values of  $K$

### Activity 3.5 Exponential growth of a human population

The population size of a country is currently 5 million, and its proportionate growth rate is 3% per year, so that  $K = 0.03 \text{ year}^{-1}$ .

- Use equation (3.7) to predict what the population size of the country will be after ten years and after a hundred years.
- After how long will the population size reach 50 million?

In each part, give your answers to three significant figures.

Solutions are given on page 48.

For obvious reasons, this model for population change is called the **exponential model**. You may recall that in Chapter B1 this name was given to the first discrete model for population change. Which of the two models is being used should be clear from the context, but if there is any possibility of confusion, then we distinguish between them by referring to either the *discrete* exponential model or the *continuous* exponential model.

As already noted, the continuous exponential model for populations has the same mathematical form as that of the radioactive decay model. It follows that, in place of equation (3.4), we now have

$$\frac{P(t+T)}{P(t)} = e^{KT}. \quad (3.8)$$

This equation says that, in a time interval of duration  $T$ , the population size changes by a factor  $e^{KT}$ , regardless of the time  $t$  at the start of the interval. If  $K < 0$ , then the population is in decline and has a half-life  $T$ , given by  $T = -(\ln 2)/K$ , just as for radioactive decay. If  $K > 0$ , then the population size increases. Here we can define the **doubling time**  $T$  for the population, which is the time that it takes for the population to double in size from any starting value.

### Activity 3.6 Doubling time

- Suppose that the growth of a population is described by a (continuous) exponential model, with proportionate growth rate  $K > 0$ . Determine, in terms of  $K$ , the doubling time for the population.
- Find the doubling time, to three significant figures, for the population described in Activity 3.5.

Solutions are given on page 48.

### Finding the parameters of the exponential model

As with the decay constant for radioactive decay, the proportionate growth rate for a population may be deduced from data on population sizes. In the case of a human population, this will be census data.

For example, the table below shows census data for the USA for the years 1790–1890. We analysed these data to an extent in Chapter B1, using the discrete exponential model. There we estimated the (discrete) proportionate growth rate by using two data points, namely, the population sizes in 1790 and 1890.

See Chapter B1,  
Subsection 2.2.

See Activity 2.2 of  
Chapter B1. See also  
Figure 2.6 of that chapter.



This has the advantage of relative simplicity, but a more accurate answer can be expected if all the available data are taken into account. We shall now do so in the context of the continuous exponential model, to estimate a value for the proportionate growth rate  $K$ .

Year	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890
Population	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4	39.8	50.2	62.9

The population is given in millions, to the nearest 100 000.

We start from the equation

$$P = P_0 e^{Kt}, \tag{3.7}$$

and take the logarithm of each side. This gives

$$\begin{aligned} \ln P &= \ln(P_0 e^{Kt}) \\ &= \ln P_0 + \ln(e^{Kt}) = \ln P_0 + Kt. \end{aligned}$$

Recall that  $\ln(e^x) = x$ , and that  $\ln(xy) = \ln x + \ln y$ .

Now compare this equation with the general form of the linear function  $y = mx + c$ , as follows:

$$\begin{array}{ccccc} y & = & mx & + & c \\ \updownarrow & & \updownarrow & & \updownarrow \\ \ln P & = & Kt & + & \ln P_0 \end{array}$$

Here  $m$  means the constant slope of a linear function, and not the mass of a sample of radioactive substance.

This comparison shows that, provided the exponential model is appropriate, we have a linear relationship between the variables  $\ln P$  and  $t$ . Hence a plot of  $\ln P$  against  $t$  should result in the data points lying on or close to a straight line which crosses the  $(\ln P)$ -axis at  $\ln P_0$  and has slope  $K$ . Such a plot of  $\ln P$  against  $t$  is called a **log-linear plot**.

Similar comparisons were carried out for the radioactive decay model, and log-linear plots were drawn from data. In that case,  $c = 0$ .

**Example 3.1 US population (1790–1890)**

- (a) By using a log-linear plot for the data in the table above, show that the exponential model is appropriate for the US population for the years 1790–1890. Measure time  $t$  in years, from  $t = 0$  at the 1790 census date.
- (b) Use your plot to estimate values for the proportionate growth rate  $K$  and the initial population size  $P_0$  in the exponential model.
- (c) Use this exponential model to predict the population of the USA in 1950.

**Solution**

- (a) The table below gives the values of the time  $t$ , the population size  $P$  and  $\ln P$ , to three significant figures.

$t$	0	10	20	30	40	50	60	70	80	90	100
$P$	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4	39.8	50.2	62.9
$\ln P$	1.36	1.67	1.97	2.26	2.56	2.84	3.14	3.45	3.68	3.92	4.14

The corresponding log-linear plot is shown in Figure 3.5.

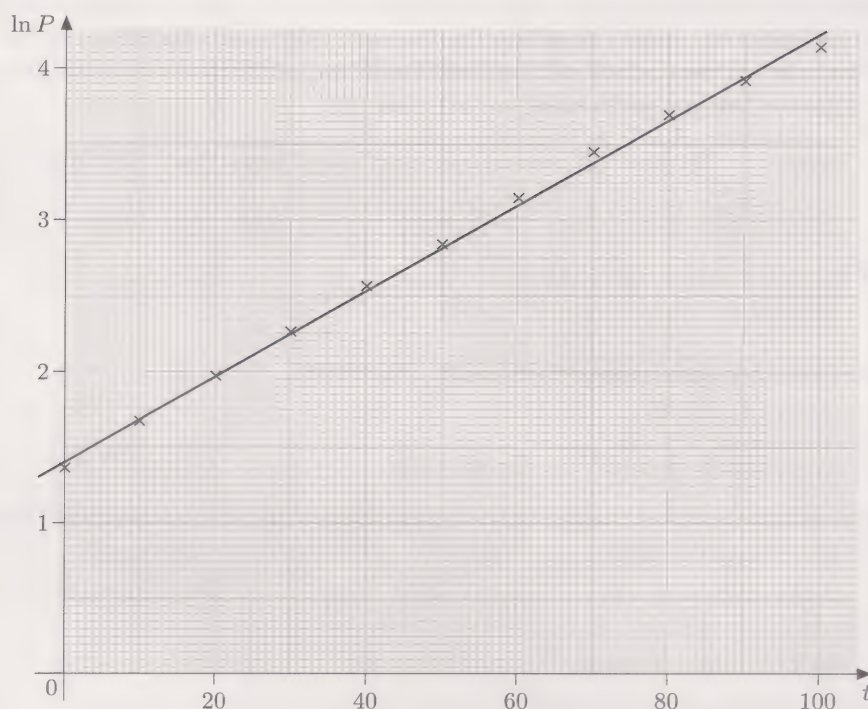


Figure 3.5 Plot of  $\ln P$  against  $t$  for the data in the table

The straight line was drawn 'by eye'.

The data points are fitted reasonably well by the straight line drawn in the figure. This confirms that the exponential model is appropriate to describe the US population for the years 1790–1890.

- (b) The intercept of the line on the  $(\ln P)$ -axis is 1.40; this is the value of  $\ln P_0$  for the model. Since  $\ln P_0 = 1.40$ , we have

$$P_0 = e^{1.40} = 4.06 \text{ (to 3 s.f.)}.$$

As well as the point  $(0, 1.40)$ , another point on the straight line is  $(100, 4.23)$ . The slope of the straight line is therefore

$$\frac{4.23 - 1.40}{100 - 0} = 0.0283.$$

So our estimate for the proportionate growth rate  $K$  is  $0.0283 \text{ year}^{-1}$ , and for the initial population size  $P_0$  the estimate is 4.06 (million).

- (c) Using the estimates obtained for  $K$  and  $P_0$ , the exponential model for the US population (1790–1890) is

$$P = 4.06e^{0.0283t}.$$

The year 1950 corresponds to  $t = 160$ , so our estimate for the population in that year is

$$P(160) = 4.06e^{0.0283 \times 160} = 376 \text{ (to 3 s.f.)}.$$

So our estimate for the population of the USA in 1950 is 376 million.

### Comment

The actual US population in 1950 was only 150.7 million! This shows that the exponential model used here cannot be expected to give good results much beyond the period 1790–1890 for which the data were given.

In the following activity, you can carry out a similar exercise for another population.



**Activity 3.7 Population of India (1950–2000)**

The table below shows data for the population of India (in millions, to the nearest million) for the years 1950–2000.

Year	1950	1960	1970	1980	1990	2000
Population	358	442	555	689	851	1014

- By using a log-linear plot for the data in this table, show that the exponential model is appropriate for the Indian population for the years 1950–2000. Measure time  $t$  in years, from  $t = 0$  at the 1950 census date.
  - Use your plot to estimate values for the proportionate growth rate  $K$  and the initial population size  $P_0$  in the exponential model.
  - Use this exponential model to predict the population of India in 2020.
- Solutions are given on page 48.

**Mathematical note**

For each of the radioactive decay and population models, we restricted the dependent variable ( $m$  and  $P$ , respectively) to be positive. In terms of the independent variable  $x$  and dependent variable  $y$ , we have derived the result that  $dy/dx = Ky$  ( $y > 0$ ) has general solution  $y = Ae^{Kx}$ , where  $A$  is a positive but otherwise arbitrary constant. The restriction on  $y$  arose from modelling considerations, though it was also relied on in the method of solution.

If the restriction on  $y$  is removed, then the expression above for the general solution remains correct, but with no restriction on  $A$ ; that is,

$$\frac{dy}{dx} = Ky \quad \text{has the general solution} \quad y = Ae^{Kx}, \quad (3.9)$$

where  $A$  is an arbitrary constant (which may be positive, negative or zero). This solution can be checked, as usual, by verifying that the differential equation is valid when  $y = Ae^{Kx}$  is substituted into it.

The method used the fact that

$$\int \frac{1}{y} dy = \ln y + c,$$

for  $y > 0$ . An alternative approach is required to deal with the cases  $y \leq 0$ .

**Comparison with the discrete exponential model**

This subsection concludes by considering how the continuous exponential model for population change compares with the corresponding discrete model developed in Chapter B1. The discrete model was given by the recurrence relation

$$P_{n+1} = (1 + r)P_n \quad (n = 0, 1, 2, \dots),$$

where  $P_n$  is the population size at  $n$  years after a specified initial time and  $r$  is the (discrete) proportionate growth rate (giving the factor by which the population increases year on year). This recurrence relation has the closed-form solution

$$P_n = (1 + r)^n P_0 \quad (n = 0, 1, 2, \dots).$$

From either the recurrence relation or its solution, we have

$$\frac{P_{n+1}}{P_n} = 1 + r. \quad (3.10)$$

The remainder of this subsection will not be assessed.

See Chapter B1, Subsection 2.2.

This can be compared with the corresponding formula for the continuous model, which is

$$\frac{P(t+T)}{P(t)} = e^{KT}. \quad (3.8)$$

Now  $P_n$  and  $P_{n+1}$  are population sizes taken one year apart, at  $t = n$  and  $t = n + 1$ . Hence we can match equations (3.8) and (3.10), for integer values of  $t$ , by putting  $t = n$  and  $T = 1$  into equation (3.8). We see then that the two models will give the same values for  $P(n) = P_n$ , provided that

$$e^K = 1 + r.$$

This equation relates the *discrete* proportionate growth rate  $r$  (measured over one year), with the *continuous* proportionate growth rate  $K$  (measured instantaneously). For a given population, these two proportionate growth rates are not quite the same. However, if the population size varies slowly, then the two values will be close to each other.

As was pointed out, a continuous model would be inappropriate for a population whose size is small. Continuous models have the apparent advantage over discrete models that they predict the population at all times, and not just annually. However, for many species the birth rate is highest in the spring whereas the death rate is highest in the winter. The continuous exponential model does not take account of this seasonal variation.

Furthermore, as discussed in Chapter B1, for very large populations the environment becomes less hospitable. Birth rates then tend to decline and mortality rates to rise, so in practice the proportionate growth rate decreases with the size of the population. There are continuous models, based on differential equations, which take account both of seasonal variations and of the decrease in proportionate growth rate with population size. However, there is not the space to study them in this course.

### Summary of Section 3

This section has introduced:

- ◇ the (continuous) exponential models for radioactive decay and for population change;
- ◇ a method for finding parameter values in these models from data, using a log-linear plot;
- ◇ the form of differential equation  $dy/dx = Ky$  (occurring in each of the above models), with general solution  $y = Ae^{Kx}$ .

For example, if  $K = 0.03$  then  $r \simeq 0.0305$ , and if  $K = 0.2$  then  $r \simeq 0.22$ .

The discrete model does, to the extent that it focuses on a 'snapshot' at annual intervals.

For example, the continuous *logistic model*, which is the analogue of the discrete model in Chapter B1, Sections 3–4, is given by the differential equation

$$\frac{dP}{dt} = KP \left( 1 - \frac{P}{E} \right),$$

where  $K$  and  $E$  are constants.



## Exercises for Section 3

### Exercise 3.1

The interest on bank deposit accounts is usually compounded annually, six-monthly or quarterly. Suppose that, in order to attract customers, a certain bank introduces a type of account on which interest is compounded *continuously* at the rate of 5% per year.

- (a) Use the fact that the proportionate rate of increase of any credit balance is  $\frac{5}{100} = 0.05$  to write down a differential equation for the credit balance  $c$  (in £) of such an account at time  $t$  (in years), during any time interval in which no deposits or withdrawals are made.
- (b) Write down the general solution of this differential equation.
- (c) A customer places £100 in such an account and makes no further deposits or withdrawals in the subsequent year. Find the corresponding particular solution of the differential equation, and hence calculate the credit balance of the account after one year.
- (d) Compare the interest received by this customer after one year with that which would be obtained at a rate of 5% per year compounded
  - (i) annually;
  - (ii) six-monthly;
  - (iii) quarterly.

Compounding interest  $n$  times per year, at a rate  $r$  per year, means that an initial credit balance  $c_0$  rises to

$$c_0 \left(1 + \frac{r}{n}\right)^n$$

after one year.

## 4 Differential equations with the computer



To study this section you will need access to your computer, together with Computer Book C.

In Section 2, you saw how to solve equations of the form  $dy/dx = f(x)g(y)$  using the method of separation of variables. There are many first-order differential equations which cannot be expressed in this form. For example,

$$\frac{dy}{dx} = x + y \quad (4.1)$$

is one such equation. More generally, we can consider differential equations of the form

$$\frac{dy}{dx} = f(x, y). \quad (4.2)$$

For each point  $(x, y)$  in the (two-dimensional) domain of  $f$ , a unique real value  $f(x, y)$  is defined.

Here  $f$  is a known function of *two* variables,  $x$  and  $y$ . In some cases such differential equations can be solved to obtain a formula (either implicit or explicit) for the solution. Such a solution is said to be *analytical*. On the other hand, it is sometimes not possible to find a formula which describes a solution. In such cases, the behaviour of functions that satisfy a differential equation can be deduced instead by *numerical* means, using a computer. You will see how this is done here.

We start by considering the graphical information which may be extracted from a first-order differential equation. A solution of equation (4.2) will have a graph which is a curve in the  $(x, y)$ -plane. For each point  $(x, y)$  that the curve passes through, the gradient of the curve has the corresponding value  $f(x, y)$ .

However, the differential equation specifies such gradient values throughout the domain  $S$ , in the  $(x, y)$ -plane, of the function  $f$ , and not just on any particular curve that passes through  $S$ . Hence the behaviour of the differential equation (as opposed to any individual solution of it) can be represented graphically if we have a means of visualising gradients at all points of the  $(x, y)$ -plane.

The association of a gradient with each point in the domain  $S$  is called a **direction field** for the differential equation. The gradient at each point can be represented satisfactorily by a short line segment that has the appropriate gradient and is centred on the point concerned. It is not a practical proposition to draw such line segments at all points of  $S$ , but we can do so for a representative grid of points. If the spacing between grid points is not too large, then this will provide a good picture of how the differential equation behaves.

The direction field for the differential equation (4.1) is represented in Figure 4.1. It can be seen that the gradients shown do correspond to the numerical values of  $f(x, y) = x + y$ . For example, the gradient is zero at the origin,  $-1$  at  $(-2, 1)$  and  $2$  at  $(1, 1)$ .



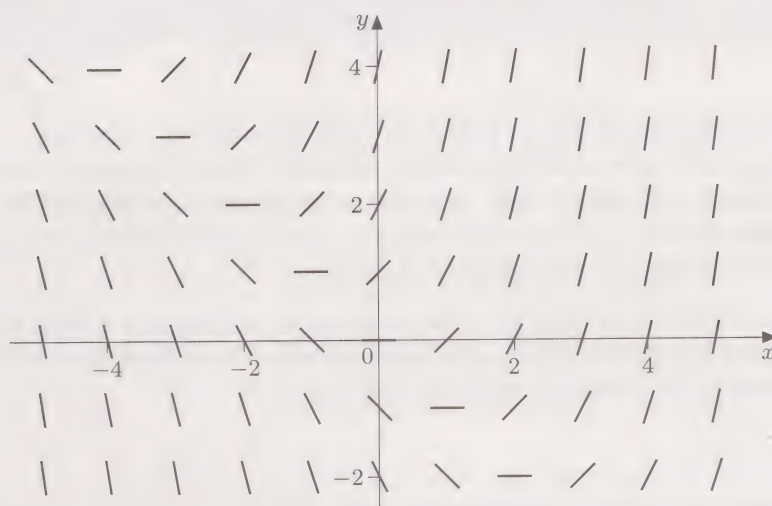


Figure 4.1 Direction field for  $dy/dx = x + y$

The graph of any solution to the differential equation must have a gradient at each point which matches that provided by the direction field. You can probably see that such graphs could readily be sketched on top of the direction field diagram, ‘following the flow’, to give a rough idea of how solutions behave. There are also more accurate ways of constructing such graphs, as you will see shortly.

*Refer to Computer Book C for the work in this section.*

You have seen how solutions to a differential equation of the type (4.2) can be obtained numerically. In fact, each such solution corresponds to a particular initial-value problem,

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

This can be solved using **Euler’s method**, which is described by the pair of recurrence relations

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n) \quad (n = 0, 1, 2, \dots).$$

In these equations,  $h$  is the **step size** between the successive values of  $x$  at which solution estimates are calculated. Each calculated value  $y_n$  is an estimate of the corresponding ‘true solution’  $y$  at  $x = x_n$ ; that is,  $y_n$  is an estimate of  $y(x_n)$ . The sequence of estimates depends on the choice of both the step size  $h$  and the overall number of steps  $N$ . Decreasing  $h$ , while increasing  $N$  to cover the same range of  $x$ -values, leads to progressively improved estimates for the solution values, and with a small enough step size, any desired level of accuracy can be achieved.

## Summary of Section 4

This section showed how first-order differential equations can be visualised graphically, using direction fields, and how solutions may be obtained either numerically or graphically, using Euler’s method.



The Swiss mathematician Leonhard Euler (1707–83) was hugely prolific. He devised the computational method that now bears his name in order to calculate the orbit of the Moon. It was used by the German cartographer and astronomer (Johann) Tobias Mayer (1723–1762) to construct lunar tables. In 1765, Mayer’s widow received £3000 from the British Board of Longitude for the contribution which his tables made to the problem of determining longitude, while Euler received £300 for his theoretical contribution to the work.

## 5 A look ahead

This section will not be assessed.

Differential equations form a vast field of study, and only the very beginnings have been described in this chapter. Their importance lies in the frequency with which they crop up as central relationships within mathematical models. You will see much more of differential equations if you undertake further studies in mathematics.

Equations of the form  $dy/dx = f(x)$  and  $dy/dx = f(x)g(y)$  do not exhaust the types of first-order differential equations for which analytical solutions can be found. For example, the differential equation

$$\frac{dy}{dx} = x + y, \quad (4.1)$$

whose direction field you studied with the computer in Section 4, has the general solution

$$y = Ae^x - x - 1,$$

where  $A$  is an arbitrary constant. Higher-order equations can also be solved analytically. For example, the second-order differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

where  $\omega$  is a non-zero constant, has the general solution

$$y = A \cos(\omega t) + B \sin(\omega t),$$

where  $A$  and  $B$  are arbitrary constants. Note the presence of *two* arbitrary constants here, for the general solution of a *second*-order equation. To obtain a particular solution in this case, two initial conditions are required, perhaps corresponding to a known position and velocity at a given time. This is typical behaviour; the number of arbitrary constants that feature in the general solution is the same as the order of the differential equation.

Where analytical solutions cannot be found, numerical methods may again be employed, for any order of differential equation. Euler's method is just the first in a long line of methods which can be used for this purpose.

We have so far considered equations that involve derivatives of a function of one independent variable, say  $y = f(x)$ . It is also possible to develop calculus for functions of two or more independent variables, say  $z = f(x, y)$ . Here there are two types of first-order derivative of  $z$  which may be taken: with respect to  $x$ , denoted  $\partial z / \partial x$ , or with respect to  $y$ , denoted  $\partial z / \partial y$ . These are called *partial* derivatives, in contrast to the *ordinary* derivatives considered in this block. Equations relating such derivatives, which again crop up within significant mathematical models, are called *partial* differential equations. As with ordinary differential equations, these may be of first or higher order. While some can be solved analytically, it is often the case that numerical methods are required to find their solutions.

As a final remark, it is possible to combine these ideas with those of vectors, which you saw in Chapter B3, and hence to arrive at the topic of *vector calculus*. Both ordinary and partial differential equations feature here.

You can check this solution by substitution.

This is called the *simple harmonic motion* equation, and provides the simplest model of an oscillating system. Again, you can check the solution by substitution.

For example,  $z$  might be the temperature at the point  $(x, y)$  of a flat metal plate.

For example, the variation in temperature  $\theta(x, t)$  at distance  $x$  along a thin rod at time  $t$  is modelled by the *heat equation*,

$$\frac{\partial^2 \theta}{\partial x^2} = k \frac{\partial \theta}{\partial t},$$

where  $k$  is a constant.



# Summary of Chapter C3

In this chapter you were introduced to first-order differential equations. You saw how to solve equations of the form  $dy/dx = f(x)$  by direct integration, and equations of the form  $dy/dx = f(x)g(y)$  by separation of variables. The more general type of equation  $dy/dx = f(x, y)$  can be solved numerically.

The continuous exponential model, embodied in the differential equation  $dy/dx = Ky$  (where  $K$  is a constant), has numerous applications. You saw it applied to radioactive decay and population change. If data are available in such a case, then values may be found for the parameters (such as  $K$ ) from a log-linear plot.

## Learning outcomes

You have been working towards the following learning outcomes.

### Terms to know and use

Differential equation, order, first-order, solution, general solution, particular solution, initial condition, initial values, initial-value problem, direct integration, implicit solution, explicit solution, implicit differentiation, separable, separation of variables, exponential growth/decay, decay constant, half-life, proportionate growth rate, doubling time, discrete/continuous exponential model, log-linear plot, direction field, Euler's method, step size.

### Notation to know and use

$y(a) = b$  (to express an initial condition on the variable  $y$ ).

### Mathematical skills

- ◇ Be able to recognise differential equations which can be solved by direct integration, and to solve suitable equations by this method.
- ◇ Be able to recognise differential equations which can be solved by separation of variables, and to solve suitable equations by this method.
- ◇ Knowing the general solution of a differential equation, be able to use an initial condition to find a particular solution.
- ◇ Know the importance of checking a solution of a differential equation by substituting it back into the equation.
- ◇ Be able to write down the general solution of a differential equation of the form  $dy/dx = Ky$ , where  $K$  is a constant.
- ◇ Given data for the variables in such a differential equation, use a log-linear plot to estimate values for the parameters (such as  $K$ ).
- ◇ In an exponential decay process, be able to find the half-life from the decay constant, or vice versa.

### Modelling skills

- ◇ Interpret predictions from an exponential model in real-world terms.

## Summary of Block C

This block has introduced calculus, which is the cornerstone of mathematical studies at this level. Facility with calculus comes at the price of much exposure to it and practice with it, so do not be perturbed if you feel that you have yet to grasp fully some of the important points.

Chapter C1 dealt with the topic of differentiation, starting with an informal explanation in terms of rates of change and the gradients of graphs, then providing a formal definition involving a limit. The consequences of this definition included a table of standard derivatives and various rules for differentiating sums and constant multiples of functions, products of functions, quotients of functions and composite functions. The alternative Leibniz notation was introduced. Differentiation was applied to the process of sketching graphs and to optimisation.

Chapter C2 started from the basis that integration is both the reverse process to differentiation and a process in its own right. A table of standard indefinite integrals was drawn up, corresponding to that for derivatives, and rules were deduced for integrating sums and constant multiples of functions. The role of basic algebraic manipulation was stressed, as a means of modifying the expression for a function until it is in a form where its integral can be recognised, from the table or otherwise. Integration was applied to analyse one-dimensional motion, and in particular motion with constant acceleration, for which specific formulas were derived connecting time, position and velocity. It was shown that definite integration provides a method for finding areas under graphs.

Chapter C3 involved in significant part revision and reinforcement of the material in the previous two chapters. Integration is central to the two strategies put forward for solving differential equations, and differentiation is needed in order to check solutions. (Also, in the shape of implicit differentiation with the Chain Rule, it permitted an explanation of why the separation of variables method works.) The principal modelling application was to exponential growth and decay, though other contexts were introduced also.

This block has been but a brief initial tour of the possibilities provided by calculus, and you can expect to come across it again on a frequent basis if you continue with mathematical studies. This will provide reinforcement to your understanding of the basic ideas and results, as well as further appreciation of its use in modelling.



# Solutions to Activities

## Solution 1.1

- (a) The differential equation is  $dy/dx = 5/(2\sqrt{x})$ .  
The derivative of the given function,  
 $y = 5\sqrt{x} = 5x^{1/2}$ , is

$$\frac{d}{dx}(5x^{1/2}) = \frac{5}{2}x^{-1/2} = \frac{5}{2\sqrt{x}},$$

which is the same expression as the right-hand side of the differential equation, as required.

- (b) The differential equation is  $dy/dx = 3e^{3x} \cos(e^{3x})$ . In order to differentiate the given function,  $y = \sin(e^{3x})$ , we apply the Composite (Chain) Rule. The function can be written as

$$y = \sin u, \quad \text{where } u = e^{3x},$$

for which

$$\frac{dy}{du} = \cos u \quad \text{and} \quad \frac{du}{dx} = 3e^{3x}.$$

Hence we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= (\cos u)(3e^{3x}) \\ &= 3e^{3x} \cos(e^{3x}), \end{aligned}$$

which is the same expression as the right-hand side of the differential equation, as required.

- (c) The differential equation is  $dy/dx = y(1 - 2 \tan(2x))$ . In order to differentiate the given function,  $y = e^x \cos(2x)$ , we apply the Product Rule. The function can be written as  $y = uv$ , where

$$u = e^x \quad \text{and} \quad v = \cos(2x),$$

for which

$$\frac{du}{dx} = e^x \quad \text{and} \quad \frac{dv}{dx} = -2 \sin(2x).$$

Hence we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx}v + u \frac{dv}{dx} \\ &= (e^x)(\cos(2x)) + (e^x)(-2 \sin(2x)) \\ &= e^x(\cos(2x) - 2 \sin(2x)). \end{aligned}$$

Now we substitute  $y = e^x \cos(2x)$  into the right-hand side of the differential equation, to obtain

$$\begin{aligned} y(1 - 2 \tan(2x)) &= e^x \cos(2x)(1 - 2 \tan(2x)) \\ &= e^x(\cos(2x) - 2 \sin(2x)). \end{aligned}$$

This is the same expression as was obtained above for the derivative, as required.

- (d) The differential equation is  $dy/dx = -\frac{1}{2}(y+1)^2$ . In order to differentiate the given function,  $y = (1-x)/(1+x)$ , we apply the Quotient Rule. The function can be written as  $y = u/v$ , where

$$u = 1 - x \quad \text{and} \quad v = 1 + x,$$

for which

$$\frac{du}{dx} = -1 \quad \text{and} \quad \frac{dv}{dx} = 1.$$

Hence we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{v^2} \left( v \frac{du}{dx} - u \frac{dv}{dx} \right) \\ &= \frac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} \\ &= -\frac{2}{(1+x)^2}. \end{aligned}$$

Now we substitute  $y = (1-x)/(1+x)$  into the right-hand side of the differential equation, to obtain

$$\begin{aligned} -\frac{1}{2}(y+1)^2 &= -\frac{1}{2} \left( \frac{1-x}{1+x} + 1 \right)^2 \\ &= -\frac{1}{2} \left( \frac{1-x+1+x}{1+x} \right)^2 \\ &= -\frac{1}{2} \left( \frac{2}{1+x} \right)^2 \\ &= -\frac{2}{(1+x)^2}. \end{aligned}$$

This is the same expression as was obtained above for the derivative, as required.

## Solution 1.2

Substitute the values  $x = 1$  and  $y = 7$  into the expression  $y = 5\sqrt{x} + c$ . This gives

$$7 = 5\sqrt{1} + c = 5 + c,$$

so  $c = 2$ . The required particular solution is therefore

$$y = 5\sqrt{x} + 2.$$

## Solution 1.3

The direct integration method can be applied to a differential equation of the form

$$\frac{d(\text{dependent variable})}{d(\text{independent variable})} = f(\text{independent variable}).$$

Equations (a), (c) and (e) are of this form, so direct integration can be applied to these. Equations (b), (d) and (f) are not of the appropriate form.

**Solution 1.4**

- (a) The general solution of
- $dy/dx = e^{2x}$
- is

$$y = \int e^{2x} dx$$

$$= \frac{1}{2}e^{2x} + c,$$

where  $c$  is an arbitrary constant.

(Check: Differentiating  $y = \frac{1}{2}e^{2x} + c$  gives  $dy/dx = e^{2x}$ , as required.)

- (b) The initial condition is
- $y = 2$
- when
- $x = 0$
- . Putting
- $x = 0$
- and
- $y = 2$
- into the general solution, we obtain

$$2 = \frac{1}{2}e^0 + c = \frac{1}{2} + c.$$

Hence  $c = \frac{3}{2}$ , and the required particular solution is

$$y = \frac{1}{2}e^{2x} + \frac{3}{2}.$$

**Solution 1.5**

The general solution of  $dx/dt = \sin t$  is

$$x = \int \sin t dt$$

$$= -\cos t + c,$$

where  $c$  is an arbitrary constant.

(Check: Differentiating  $x = -\cos t + c$  gives  $dx/dt = \sin t$ , as required.)

The initial condition is  $x = 1$  when  $t = 0$ . Putting  $t = 0$  and  $x = 1$  into the general solution, we obtain

$$1 = -\cos 0 + c = -1 + c.$$

Hence  $c = 2$ , and the solution of the initial-value problem is

$$x = 2 - \cos t.$$

**Solution 1.6**

- (a) The general solution of
- $dm/dt = -8t$
- is

$$m = \int (-8t) dt$$

$$= -4t^2 + c,$$

where  $c$  is an arbitrary constant. (This general solution can be checked by differentiation.)

- (b) The initial condition is
- $m = 200$
- when
- $t = 0$
- . Putting
- $t = 0$
- and
- $m = 200$
- into the general solution, we obtain

$$200 = 0 + c.$$

Hence  $c = 200$ , and the required particular solution is

$$m = 200 - 4t^2.$$

- (c) When
- $m = 100$
- , the corresponding value of
- $t$
- is given by

$$100 = 200 - 4t^2; \quad \text{that is, } t^2 = 25.$$

Hence  $t = 5$  when  $m = 100$ . The rocket's fuel is exhausted after 5 seconds.

**Solution 2.1**

- (a) The given equation is of the form
- $H(y) = F(x)$
- , with
- $H(y) = y + \sin y$
- and
- $F(x) = x + e^x - 1$
- . It follows from result (2.3) that

$$\frac{d}{dy}(y + \sin y) \frac{dy}{dx} = \frac{d}{dx}(x + e^x - 1),$$

which leads to

$$(1 + \cos y) \frac{dy}{dx} = 1 + e^x.$$

Dividing through by  $1 + \cos y$  (which is non-zero for  $-\pi < y < \pi$ ) gives

$$\frac{dy}{dx} = \frac{1 + e^x}{1 + \cos y},$$

as required.

- (b) At
- $(0, 0)$
- , the slope of the curve is

$$\frac{dy}{dx} = \frac{1 + e^0}{1 + \cos 0} = \frac{2}{2} = 1.$$

**Solution 2.2**

The given equation is of the form  $H(y) = F(x)$ , with  $H(y) = \tan y$  and  $F(x) = x$ . It follows from result (2.3) that

$$\frac{d}{dy}(\tan y) \frac{dy}{dx} = \frac{d}{dx}(x),$$

which leads to

$$\sec^2 y \frac{dy}{dx} = 1.$$

Dividing through by  $\sec^2 y$  (which is non-zero for  $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$ ), and using the trigonometric identity  $\sec^2 y = 1 + \tan^2 y$ , we have

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}.$$

However,  $\tan y = x$ , and so

$$\frac{dy}{dx} = \frac{1}{1 + x^2},$$

as required.



### Solution 2.3

- (a) The differential equation is  $(1/y^2)dy/dx = -1$ . Comparing with result (2.4), we have  $h(y) = 1/y^2 = y^{-2}$  and  $f(x) = -1$ . Hence the general solution of the differential equation is given by

$$\int y^{-2} dy = \int (-1) dx;$$

that is,

$$-\frac{1}{y} = -x + c,$$

where  $c$  is an arbitrary constant.

- (b) The explicit solution is obtained by making  $y$  the subject of the above equation. In this case, that is achieved by taking the reciprocal of each side and multiplying through by  $-1$ . The outcome is

$$y = \frac{1}{x - c},$$

where  $c$  is an arbitrary constant. (Since  $y > 0$ , we must have  $x - c > 0$ . There is some discussion of such restrictions on variables later in the section.)

This answer could equally well be written as

$$y = \frac{1}{x + c},$$

where  $c$  is an arbitrary constant (in which case, we must have  $x + c > 0$ ).

### Solution 2.4

Equations (a), (b), (c) and (e) are of the appropriate form, but equations (d) and (f) are not.

- (a)  $f(x) = x$ ,  $g(y) = \cos y$   
 (b)  $f(t) = t \cos t$ ,  $g(x) = 1$   
 (c)  $f(t) = 1$ ,  $g(p) = p \cos p$   
 (e)  $f(y) = 1 + y$ ,  $g(v) = v$

These are the most likely choices for  $f$  and  $g$ , but they are not the only correct possibilities. For equation (a), for example, you could have chosen  $f(x) = 2x$  and  $g(y) = \frac{1}{2} \cos y$ .

Equation (b) can also be solved using direct integration; this is the most direct method of solution.

### Solution 2.5

- (a) The differential equation  $dy/dx = -x/y$  has a right-hand side of the form  $f(x)g(y)$ , where  $f(x) = -x$  and  $g(y) = 1/y$ . Dividing both sides of the differential equation by  $g(y)$  means multiplying both sides by  $y$ . This gives

$$y \frac{dy}{dx} = -x.$$

(Other choices for the functions  $f$  and  $g$  are possible, but all choices lead to the same general solution.)

From this we obtain, using result (2.4),

$$\int y dy = \int (-x) dx.$$

(The solution thus far can be shortened by using the shortcut described in the text before Activity 2.4 on page 20. In that case, the solution to this point would simply read:

Separating the variables in  $dy/dx = -x/y$ , we obtain

$$\int y dy = \int (-x) dx.)$$

Carrying out the two integrations, we have

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c,$$

where  $c$  is an arbitrary constant. This implicit form of the general solution can be manipulated to make  $y$  the subject, giving

$$y = \sqrt{2c - x^2},$$

or equivalently

$$y = \sqrt{b - x^2},$$

where  $b = 2c$  is another arbitrary constant. We take the positive square root for  $y$  in order to satisfy the restriction  $y > 0$  given in the question.

(This expression for the general solution could be checked by substituting it into the differential equation.)

- (b) The initial condition is  $y = 1$  when  $x = 0$ . Putting  $x = 0$  and  $y = 1$  into the general solution, we have

$$1 = \sqrt{b - 0^2} = \sqrt{b}.$$

Hence  $b = 1$ , and the required particular solution is

$$y = \sqrt{1 - x^2}.$$

(For this expression to be well-defined, the range of values of  $x$  must be restricted. We shall return to this problem later in the section.)

### Solution 2.6

- (a) The shortcut is adopted here. The differential equation is  $dx/dt = t \sec x$ , so we can choose  $f(t) = t$  and  $g(x) = \sec x$ . Separating the variables in the differential equation, we obtain

$$\int \frac{1}{\sec x} dx = \int t dt;$$

that is,

$$\int \cos x dx = \int t dt.$$

After integration, we have

$$\sin x = \frac{1}{2}t^2 + c,$$

where  $c$  is an arbitrary constant. Since  $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ , making the dependent variable  $x$  the subject of the equation, we obtain

$$x = \arcsin(\frac{1}{2}t^2 + c)$$

for the general solution in explicit form.

- (b) The initial condition is  $x = 0$  when  $t = 0$ . Putting  $t = 0$  and  $x = 0$  into the general solution gives

$$0 = \arcsin(0 + c) = \arcsin c.$$

Hence  $c = 0$ , and the required particular solution is

$$x = \arcsin(\frac{1}{2}t^2).$$

### Solution 2.7

- (a) Separation of variables in the differential equation gives

$$\int v dv = \int \left(-\frac{k}{r^2}\right) dr.$$

Hence we have

$$\frac{1}{2}v^2 = \frac{k}{r} + c,$$

or equivalently (since  $v > 0$ )

$$v = \sqrt{2\left(\frac{k}{r} + c\right)},$$

where  $c$  is an arbitrary constant. This is the general solution of the differential equation.

- (b) The initial condition is  $v(R) = v_R$ . Putting  $r = R$  and  $v = v_R$  into the general solution, we have

$$v_R = \sqrt{2\left(\frac{k}{R} + c\right)}.$$

Hence  $c = \frac{1}{2}v_R^2 - k/R$ , and the required particular solution is

$$v = \sqrt{v_R^2 + 2k\left(\frac{1}{r} - \frac{1}{R}\right)}.$$

- (c) As  $r$  becomes large,  $1/r \rightarrow 0$ , so the velocity

$$v = \sqrt{v_R^2 + 2k\left(\frac{1}{r} - \frac{1}{R}\right)}$$

tends towards the limiting value

$$\sqrt{v_R^2 - \frac{2k}{R}}.$$

If  $v_R > \sqrt{2k/R}$  then  $v_R^2 > 2k/R$ , and so

$$\sqrt{v_R^2 - \frac{2k}{R}} > 0.$$

Therefore, if  $v_R > \sqrt{2k/R}$ , the probe continues to travel away from the Earth indefinitely, since its velocity is never zero.

On the other hand, if  $v_R < \sqrt{2k/R}$  then  $v_R^2 < 2k/R$ , and so no real value for the velocity is obtained at sufficiently large distances from the Earth. In this case the probe does not travel that far away; it attains a maximum distance from the Earth and then falls back.

(An alternative approach to this question would be to examine where the velocity of the probe becomes zero. This occurs where

$$r = \left(\frac{1}{R} - \frac{v_R^2}{2k}\right)^{-1},$$

which gives a positive value for  $r$  only if  $v_R < \sqrt{2k/R}$ . Hence the velocity of the probe is never zero if  $v_R > \sqrt{2k/R}$ .)

The critical launch velocity,  $v_R = \sqrt{2k/R}$ , is called the Earth's *escape velocity*. Its significance is that, according to the model, a probe launched with this or a greater velocity will escape completely from the Earth, whereas a probe launched with a lesser velocity will eventually fall back towards the Earth. (The value of the Earth's escape velocity is approximately  $1.12 \times 10^4 \text{ m s}^{-1}$ , that is, about 40 000 kph or 25 000 mph.)

### Solution 3.1

- (a) The differential equation is  $dm/dt = -km$  ( $m > 0$ ). Separating the variables, we obtain

$$\int \frac{1}{m} dm = - \int k dt.$$

Since  $m > 0$ , this gives

$$\ln m = -kt + c,$$

where  $c$  is an arbitrary constant. This is the general solution in implicit form.



Taking the exponential of each side, we obtain

$$\begin{aligned} m &= e^{-kt+c} \\ &= e^c e^{-kt} = Ae^{-kt}, \end{aligned}$$

where  $A = e^c$  is a positive but otherwise arbitrary constant. This is the general solution in explicit form.

- (b) The initial condition is  $m = m_0$  when  $t = 0$ . Putting  $t = 0$  and  $m = m_0$  into the general solution, we have

$$m_0 = Ae^0 = A.$$

Hence  $A = m_0$ , and the required particular solution is

$$m = m_0 e^{-kt}.$$

### Solution 3.2

Using equation (3.5), the decay constant  $k$  corresponding to a half-life  $T = 2.62$  hours is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{2.62} = 0.265 \text{ hour}^{-1} \text{ (to 3 s.f.)}.$$

Hence, from equation (3.4), the proportion of the original amount remaining after six hours is

$$\frac{m(t+6)}{m(t)} = e^{-0.265 \times 6} = 0.204 \text{ (to 3 s.f.)}.$$

So about 20% of the original amount of silicon-31 remains after six hours.

### Solution 3.3

Using equation (3.5), the decay constant  $k$  corresponding to a half-life  $T = 5570$  years is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5570} = 1.24 \times 10^{-4} \text{ year}^{-1} \text{ (to 3 s.f.)}.$$

At the present time, 85% of the original amount  $m_0$  of carbon-14 remains, so we have

$$0.85 = \frac{m}{m_0} = e^{-kt},$$

where  $t$  (in years) is the time since the animal died. Taking the logarithm of each side, we obtain

$$-kt = \ln(0.85);$$

that is,

$$\begin{aligned} t &= -\frac{\ln(0.85)}{k} \\ &= -\frac{\ln(0.85)}{1.24 \times 10^{-4}} = 1310 \text{ years (to 3 s.f.)}. \end{aligned}$$

So the animal died approximately 1300 years ago.

### Solution 3.4

- (a) The completed table of data is below.

$t$ (min)	10	20	30	40	50
$m/m_0$	0.73	0.56	0.42	0.29	0.24
$\ln(m/m_0)$	-0.31	-0.58	-0.87	-1.24	-1.43

- (b) The graph of  $\ln(m/m_0)$  against  $t$  for the data in part (a) is shown in Figure S.1. This figure also shows a straight line (chosen by eye) through the origin which fits the data points well. (This good fit confirms that the radioactive decay of uranium-239 can be modelled satisfactorily as an exponential decay process.)

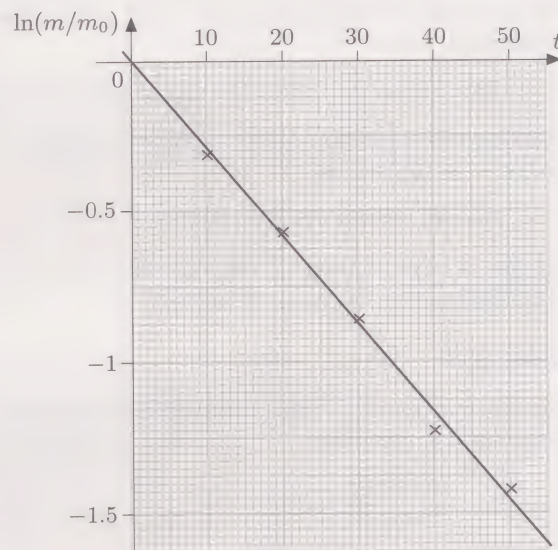


Figure S.1

- (c) The straight line passes through the origin and the point  $(50, -1.48)$ , so its slope is

$$-\frac{1.48}{50} = -0.0296.$$

Hence the decay constant for uranium-239 is  $k = 0.0296 \text{ min}^{-1}$ . From equation (3.5), the corresponding half-life is

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.0296} = 23.4 \text{ min (to 3 s.f.)}.$$

Solution 3.5

- (a) The initial population size is  $P_0 = 5 \times 10^6$ , and the proportionate growth rate is  $K = 0.03 \text{ year}^{-1}$ . Hence, from equation (3.7), the population size  $P$  after time  $t$  years is

$$P = P_0 e^{Kt} = 5 \times 10^6 e^{0.03t}.$$

After 10 years the population size is

$$P = 5 \times 10^6 e^{0.3} = 6.75 \times 10^6 \text{ (to 3 s.f.)},$$

whereas after 100 years, the population size is

$$P = 5 \times 10^6 e^3 = 1.00 \times 10^8 \text{ (to 3 s.f.)}.$$

So, to three significant figures, the population size after 10 years is 6.75 million, and after 100 years it is 100 million.

- (b) The population size reaches  $50 \times 10^6$  at the time  $t$  years given by

$$50 \times 10^6 = 5 \times 10^6 e^{0.03t}; \text{ that is, } e^{0.03t} = 10.$$

Taking the logarithm of each side, we obtain  $0.03t = \ln 10$ ; that is,

$$t = \frac{100}{3} \ln 10 = 76.8 \text{ (to 3 s.f.)}.$$

Hence it takes almost 77 years for the population size to reach 50 million.

Solution 3.6

- (a) From equation (3.8), the population will double in a time  $T$  given by

$$2 = \frac{P(t+T)}{P(t)} = e^{KT}.$$

Taking the logarithm of each side, we obtain  $KT = \ln 2$ , so the doubling time is

$$T = \frac{\ln 2}{K}.$$

- (b) For the population described in Activity 3.5, the proportionate growth rate is  $K = 0.03 \text{ year}^{-1}$ . Hence the doubling time is

$$T = \frac{\ln 2}{0.03} = 23.1 \text{ years (to 3 s.f.)}.$$

Solution 3.7

- (a) The table below gives the values of the time  $t$  (in years since 1950), the population size  $P$  (in millions) and  $\ln P$ , to three significant figures.

$t$	0	10	20	30	40	50
$P$	358	442	555	689	851	1014
$\ln P$	5.88	6.09	6.32	6.54	6.75	6.92

The corresponding log-linear plot is shown in Figure S.2.

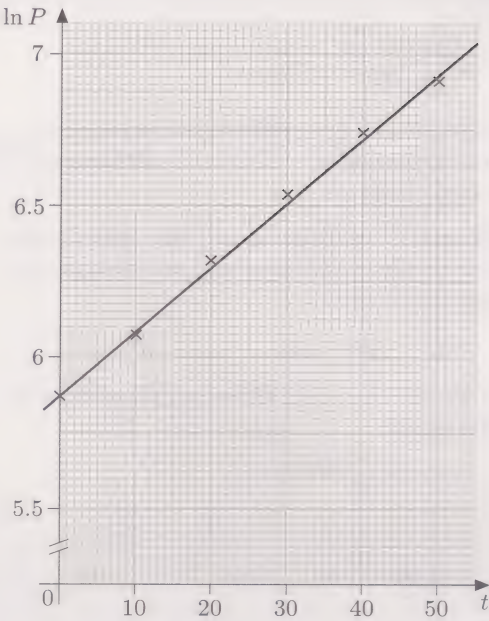


Figure S.2

The data points are fitted reasonably well by the straight line drawn. This confirms that the exponential model is appropriate to describe the Indian population for the years 1950–2000.

- (b) The intercept on the  $(\ln P)$ -axis is 5.89; this is the value of  $\ln P_0$  for the model. Since  $\ln P_0 = 5.89$ , we have

$$P_0 = e^{5.89} = 361 \text{ (to 3 s.f.)}.$$

As well as the point  $(0, 5.89)$ , another point on the straight line is  $(50, 6.94)$ . The slope of the straight line is therefore

$$\frac{6.94 - 5.89}{50 - 0} = 0.021.$$

So our estimate for the proportionate growth rate  $K$  is  $0.021 \text{ year}^{-1}$ , and for the initial population  $P_0$  the estimate is 361 (million).

- (c) Using the estimates obtained for  $K$  and  $P_0$ , the exponential model for the Indian population (1950–2000) is

$$P = 361e^{0.021t}.$$

The year 2020 corresponds to  $t = 70$ , so our estimate for the population in that year is

$$P(70) = 361e^{0.021 \times 70} = 1570 \text{ (to 3 s.f.)}.$$

So our estimate for the population of India in 2020 is 1570 million, that is, 1.57 billion.



# Solutions to Exercises

## Solution 1.1

The differential equation is  $dy/dx = y^2$ . The derivative of the given function,  $y = -1/x$ , is

$$\frac{d}{dx} \left( -\frac{1}{x} \right) = -\frac{d}{dx} (x^{-1}) = -(-x^{-2}) = \frac{1}{x^2}.$$

Now we substitute  $y = -1/x$  into the right-hand side of the differential equation, to obtain

$$y^2 = \left( -\frac{1}{x} \right)^2 = \frac{1}{x^2}.$$

This is the same expression as was obtained above for the derivative. Hence  $y = -1/x$  ( $x > 0$ ) is a solution of the given differential equation.

## Solution 1.2

(a) The general solution of  $dy/dx = \sqrt{x} = x^{1/2}$  is

$$\begin{aligned} y &= \int x^{1/2} dx \\ &= \frac{2}{3} x^{3/2} + c, \end{aligned}$$

where  $c$  is an arbitrary constant.

(b) The initial condition is  $y = 5$  when  $x = 4$ . Putting  $x = 4$  and  $y = 5$  into the general solution, we obtain

$$5 = \frac{2}{3} \times 4^{3/2} + c = \frac{16}{3} + c.$$

Hence  $c = -\frac{1}{3}$ , and the required particular solution is

$$y = \frac{2}{3} x^{3/2} - \frac{1}{3}.$$

(c) When  $x = 25$ , we have

$$\begin{aligned} y &= \frac{2}{3} \times 25^{3/2} - \frac{1}{3} \\ &= \frac{2}{3} \times 125 - \frac{1}{3} \\ &= \frac{1}{3}(250 - 1) = 83. \end{aligned}$$

## Solution 1.3

(a) The general solution of  $du/dx = \cos(2x)$  is

$$\begin{aligned} u &= \int \cos(2x) dx \\ &= \frac{1}{2} \sin(2x) + c, \end{aligned}$$

where  $c$  is an arbitrary constant. Using the initial condition,  $u = -2$  when  $x = \frac{1}{4}\pi$ , we obtain

$$-2 = \frac{1}{2} \sin\left(\frac{1}{2}\pi\right) + c = \frac{1}{2} + c.$$

Hence  $c = -\frac{5}{2}$ , and the solution of the initial-value problem is

$$u = \frac{1}{2} \sin(2x) - \frac{5}{2}.$$

(b) The general solution of  $dx/dt = 1/t$  ( $t > 0$ ) is

$$\begin{aligned} x &= \int \frac{1}{t} dt \\ &= \ln t + c, \end{aligned}$$

where  $c$  is an arbitrary constant. Using the initial condition,  $x = 3$  when  $t = 1$ , we obtain

$$3 = \ln 1 + c = c.$$

Hence  $c = 3$ , and the solution of the initial-value problem is

$$x = \ln t + 3.$$

## Solution 1.4

(a) The general solution of the differential equation  $dr/dt = -k$  is

$$\begin{aligned} r &= -\int k dt \\ &= -kt + c, \end{aligned}$$

where  $c$  is an arbitrary constant.

(b) Since the initial radius of the snowball is 100 cm, the initial condition is  $r = 100$  when  $t = 0$ . Substituting this into the general solution, we obtain  $c = 100$ . Hence the required particular solution is

$$r = 100 - kt.$$

(c) The initial volume of the snowball (in  $\text{cm}^3$ ) is  $\frac{4}{3}\pi \times 100^3$ . If we denote the radius of the snowball after 2 days by  $R$ , then

$$\frac{4}{3}\pi R^3 = \frac{1}{2} \times \frac{4}{3}\pi 100^3,$$

from which we find

$$R = 2^{-1/3} \times 100.$$

Using the condition  $r = R = 2^{-1/3} \times 100$  at  $t = 2$  in the particular solution of the differential equation, we obtain

$$2^{-1/3} \times 100 = 100 - 2k.$$

It follows that

$$k = 50(1 - 2^{-1/3}) \simeq 10.315.$$

(d) The snowball disappears when  $r = 0$ ; that is, when  $100 - kt = 0$ . This occurs when

$$t = \frac{100}{k} = \frac{2}{1 - 2^{-1/3}} \simeq 9.695.$$

It therefore takes about 9.7 days for the snowball to disappear.

**Solution 2.1**

- (a) The given equation is of the form  $H(y) = F(x)$ , with  $H(y) = y + \ln y$ , where  $y > 0$ , and  $F(x) = (1 - 2x)e^{3x}$ . It follows from result (2.3) that

$$\frac{d}{dy}(y + \ln y) \frac{dy}{dx} = \frac{d}{dx}((1 - 2x)e^{3x}),$$

which (using the Product Rule on the right-hand side) leads to

$$\left(1 + \frac{1}{y}\right) \frac{dy}{dx} = (-2)(e^{3x}) + (1 - 2x)(3e^{3x});$$

that is,

$$\frac{y+1}{y} \frac{dy}{dx} = (1 - 6x)e^{3x}.$$

Multiplying through by  $y/(y+1)$  gives

$$\frac{dy}{dx} = \frac{y(1 - 6x)e^{3x}}{y+1},$$

as required.

- (b) At  $(0, 1)$ , the slope of the curve is

$$\frac{dy}{dx} = \frac{1(1 - 0)e^0}{1 + 1} = \frac{1}{2}.$$

**Solution 2.2**

- (a) The differential equation is  $u^2 du/dx = x^4$ . Comparing with result (2.4), we have  $h(u) = u^2$  and  $f(x) = x^4$ . Hence the general solution of the differential equation is given by

$$\int u^2 du = \int x^4 dx;$$

that is,

$$\frac{1}{3}u^3 = \frac{1}{5}x^5 + c,$$

where  $c$  is an arbitrary constant. Solving this equation for  $u$  gives the explicit form of the general solution as

$$u = \left(\frac{3}{5}x^5 + b\right)^{1/3},$$

where  $b = 3c$  is an arbitrary constant.

- (b) The initial condition is  $u = 1$  when  $x = 1$ . Putting  $x = 1$  and  $u = 1$  into the general solution, we obtain

$$1 = \left(\frac{3}{5} + b\right)^{1/3}.$$

Hence  $b = \frac{2}{5}$ , and the required particular solution is

$$u = \left(\frac{3}{5}x^5 + \frac{2}{5}\right)^{1/3}.$$

**Solution 2.3**

- (a) The differential equation is  $dy/dx = -y^3$  ( $y > 0$ ). Separating the variables, we obtain

$$\int \left(-\frac{1}{y^3}\right) dy = \int 1 dx;$$

that is,

$$\frac{1}{2y^2} = x + c,$$

where  $c$  is an arbitrary constant. To make  $y$  the subject of this equation, first take the reciprocals of both sides and then divide through by 2, to obtain

$$y^2 = \frac{1}{2(x + c)}.$$

Since  $y > 0$ , the general solution in explicit form is therefore

$$y = \frac{1}{\sqrt{2(x + c)}}.$$

The initial condition is  $y = \frac{1}{2}$  when  $x = 0$ .

Putting  $x = 0$  and  $y = \frac{1}{2}$  into the general solution, we obtain

$$\frac{1}{2} = \frac{1}{\sqrt{2c}}.$$

Hence  $c = 2$ , and the required particular solution is

$$y = \frac{1}{\sqrt{2(x + 2)}}.$$

- (b) The differential equation is  $dx/dt = tx$  ( $x > 0$ ). Separating the variables, we obtain

$$\int \frac{1}{x} dx = \int t dt;$$

that is,

$$\ln x = \frac{1}{2}t^2 + c,$$

where  $c$  is an arbitrary constant. To make  $x$  the subject of this equation, recall that  $\exp(\ln x) = x$ . Hence, taking exponentials of both sides, we have

$$\begin{aligned} x &= e^{(t^2/2)+c} \\ &= e^c e^{t^2/2} = Ae^{t^2/2}, \end{aligned}$$

where  $A = e^c$ . The constant  $A$  must be positive (since  $e^c > 0$  for any choice of  $c$ ) but is otherwise arbitrary. The general solution for  $x > 0$ , in explicit form, is therefore

$$x = Ae^{t^2/2},$$

where  $A$  is a positive but otherwise arbitrary constant.

The initial condition is  $x = 2$  when  $t = 0$ .

Putting  $t = 0$  and  $x = 2$  into the general solution, we have

$$2 = Ae^0.$$

Hence  $A = 2$ , and the required particular solution is

$$x = 2e^{t^2/2}.$$



### Solution 2.4

The volume  $V$  is decreasing, and the rate of change of volume  $dV/dt$  is proportional to the surface area  $A$ . In other words,

$$\frac{dV}{dt} = -kA,$$

where  $k$  is the (positive) constant of proportionality. Putting  $V$  and  $A$  in terms of  $r$ , this differential equation becomes

$$\frac{d}{dt} \left( \frac{4}{3} \pi r^3 \right) = -4\pi k r^2.$$

Hence, using the Chain Rule, we have

$$4\pi r^2 \frac{dr}{dt} = -4\pi k r^2,$$

which gives

$$\frac{dr}{dt} = -k.$$

### Solution 2.5

- (a) The differential equation is  $dh/dt = -kh^{1/2}$  ( $h > 0$ ). Separating the variables, we obtain

$$\int h^{-1/2} dh = \int (-k) dt;$$

that is,

$$2h^{1/2} = -kt + c,$$

where  $c$  is an arbitrary constant. Hence the explicit form of the general solution is

$$h = \frac{1}{4}(c - kt)^2.$$

- (b) The initial condition is  $h = h_0$  when  $t = 0$ . Putting  $t = 0$  and  $h = h_0$  into the general solution, we obtain

$$h_0 = \frac{1}{4}c^2.$$

Now the equation  $2h^{1/2} = -kt + c$  above shows that  $c$  must be positive, since we have  $h^{1/2} > 0$ ,  $k > 0$  and  $t > 0$ . Hence  $c = 2h_0^{1/2}$ , and the required particular solution is

$$h = \left( h_0^{1/2} - \frac{1}{2}kt \right)^2.$$

- (c) The tank empties to the level of the hole when  $h = 0$ . This occurs when

$$h_0^{1/2} - \frac{1}{2}kt = 0;$$

that is, at time

$$t = \frac{2h_0^{1/2}}{k}.$$

### Solution 3.1

- (a) The credit balance of the account increases at a *proportionate* rate 0.05, so the balance  $c$  at time  $t$  satisfies the differential equation

$$\frac{dc}{dt} = 0.05c.$$

- (b) By comparison with result (3.9), the general solution is

$$c = Ae^{0.05t},$$

where  $A$  is an arbitrary constant. (However, we require  $A > 0$  for the purposes of the model.)

- (c) If the initial balance is £100, then the initial condition is  $c = 100$  when  $t = 0$ . Substituting these values into the general solution, we obtain

$$100 = Ae^0 = A,$$

so the required particular solution is

$$c = 100e^{0.05t}.$$

The credit balance after one year is therefore

$$c(1) = 100e^{0.05} = 105.13,$$

to the nearest penny. At the end of one year, the balance of the account is £105.13.

- (d) In this part we use the formula given in the margin note, with  $c_0 = 100$ .

The account balances (in £) after one year, for an interest rate of 5% per year ( $r = 0.05$ ) compounded annually ( $n = 1$ ), six-monthly ( $n = 2$ ) and quarterly ( $n = 4$ ), are respectively as follows:

$$100 \times 1.05 = 105;$$

$$100 \times (1.025)^2 = 105.06;$$

$$100 \times (1.0125)^4 = 105.09.$$

The interest accrued on £100 after one year at 5% interest is therefore £5 (compounded annually), £5.06 (compounded six-monthly), £5.09 (compounded quarterly) or £5.13 (compounded continuously).



# Index

- analytical solution 38
- carbon dating 28
- continuous exponential model 32
- decay constant 26
  - estimation 29
- differential equation
  - first-order 6
  - separable 19
- direct integration 11, 12
- direction field 38
- discrete exponential model 32
- doubling time 32
- Euler's method 39
- explicit solution 15
- exponential model 32
- function of two variables 38
- general solution of a differential equation 9
- half-life 28
- implicit differentiation 16
- implicit solution 15
- initial condition 10
- initial value 10
- initial-value problem 10
- log-linear plot 33
- order of a differential equation 6
- particular solution of a differential equation 9
- proportionate growth rate 31
- separation of variables 19
- solution curve 13
- solution of a differential equation 6
- step size 39









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